> ⓘ Info
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Using ergodic current on $\partial^2\mathbb{R}\boldsymbol{H}^n$

> **(Sullivan's measurable rigidity theorem)** Let $\Gamma_1, \Gamma_2 <_{\mathsf{d}} \operatorname{Isom}(\mathbb{R}\boldsymbol{H}^n)$ be discrete subgroups such that $\delta(\Gamma_1) = \delta(\Gamma_2) = \delta$,
>
> $$\varphi : \Gamma_1 \to \Gamma_2$$
>
> be a group homomorphism and
>
> $$\Phi : (\Lambda(\Gamma_1), \nu_1) \to (\Lambda(\Gamma_2), \nu_2)$$
>
> is a Borel bi-measurable non-singular map φ-conjugating the actions
>
> $$\forall \gamma \in \Gamma_1, \Phi \circ \gamma = \phi(\gamma) \circ \Phi$$
>
> If Γ_1 is divergent at δ and $\Lambda(\Gamma_1)$ is "large enough" then there exists a $f \in \operatorname{Isom}(\mathbb{R}\boldsymbol{H}^n)$ that conjugates Γ_1, Γ_2 and
>
> $$\Phi = f \ \nu_1\text{-ae}$$

> **Proposition:** Let A be a μ-full subset. Then
>
> $$\operatorname{supp}(\mu) \subseteq \overline{A}$$

- Let $x \in \operatorname{supp}(\mu)$, then

  $$\forall \text{ onb } U_x, \mu(U_x) > 0$$

  which means

  $$\cancelto{0}{\mu(P^c \cap U_x)} + \mu(P \cap U_x) = \mu(U_x) > 0$$

  thus, every onb U_x of x intersects A, implying

  $$\operatorname{supp}(\mu) \subseteq \overline{A}$$

> **(Sullivan's preservation of cross ratios)** Let X_1, X_2 be two Riemannian manifolds of sectional curvature ≤ -1 and let
>
> $$\varphi : \Gamma_1 \to \Gamma_2$$
>
> with $\delta(\Gamma_1) = \delta(\Gamma_2) = \delta$ is a group homomorphism and
>
> $$\Phi : (\Lambda(\Gamma_1), \nu_1) \to (\Lambda(\Gamma_2), \nu_2)$$
>
> is a Borel bi-measurable non-singular map φ-conjugating the actions
>
> $$\forall \gamma \in \Gamma_1, \Phi \circ \gamma = \phi(\gamma) \circ \Phi$$
>
> If
>
> $$(\Phi \times \Phi)^* \mu_2 = c\mu_1$$
>
> then Φ preserves cross-ratios ν_1-almost everywhere on $\Lambda(\Gamma_1)$.

- There is a $f \in \operatorname{Moeb}(\partial\mathbb{R}\boldsymbol{H}^n)$ such that

  $$\Phi|_P = f|_P$$

  on a full measure $P \subseteq \Lambda(\Gamma_1)$.
  - By

    > **Proposition:** Let A be a μ-full subset. Then
    >
    > $$\operatorname{supp}(\mu) \subseteq \overline{A}$$

    $$\begin{aligned} &\overline{P} = \Lambda(\Gamma_1) \\ \implies f(\Lambda(\Gamma_1)) = f(\overline{P}) = &\overline{f(P)} = \Lambda(\Gamma_2) \end{aligned}$$

  - Therefore,

    $$f : \Lambda(\Gamma_1) \cong \Lambda(\Gamma_2)$$

- Also, we have

  $$f|_P \circ \gamma|_P = \phi(\gamma) \circ f|_P$$

  the on a full measure $P \subseteq \Lambda(\Gamma_1)$, but these are continuous functions so they must agree on the closure $\overline{P} = \Lambda(\Gamma_1)$ as well

  $$f|_{\Lambda(\Gamma_1)} \circ \gamma|_{\Lambda(\Gamma_1)} = \phi(\gamma) \circ f|_{\Lambda(\Gamma_1)}$$

  - Therefore, the action on the limit sets are φ-equivariant Mobius-conjugate.
  - If the limit set $\Lambda(\Gamma_1)$ contains $n+1$ points in general position, then

    $$f \circ \gamma = \phi(\gamma) \circ f$$

    in $\operatorname{Moeb}(\partial\mathbb{R}\boldsymbol{H}^n)$.

- Let Γ_1, Γ_2 be such that,

  $$\begin{aligned} \Lambda(\Gamma_1) = \Lambda(\Gamma_2) &=: \Lambda \\ \text{convex-span}(\Lambda) &= \partial\mathbb{R}\boldsymbol{H}^m \\ \varphi : \Gamma_1 &\cong_{\mathsf{Grp}} \Gamma_2 \end{aligned}$$

then $\forall \gamma_1 \in \Gamma_1$

$$\begin{aligned} &\gamma_1|_\Lambda = \varphi(\gamma_1)|_\Lambda \\ \implies &\gamma_1 \equiv \varphi(\gamma_1) \end{aligned}$$