> Info
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Counting orbits of discrete subgroups of $O^+(1,n)(\mathbb{R})$ in $\mathbb{R}\boldsymbol{H}^n$

> Let
> $$\Gamma \leq O^+(1,n)(\mathbb{R}) \cong \mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n) \curvearrowright \mathbb{R}\boldsymbol{H}^n$$
> be a discrete subgroup.

We know

$$\Gamma \curvearrowright \mathbb{R}\boldsymbol{H}^n$$

is a proper action so $|\Gamma_x| < \infty$ which means the orbit map

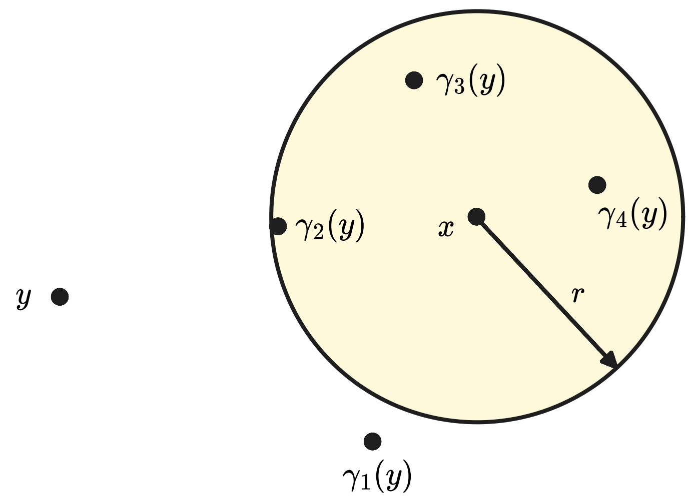
$$\begin{aligned} \Gamma &\to \mathbb{R}\boldsymbol{H}^n \\ \gamma &\mapsto \gamma(y) \end{aligned}$$

gives a bijection

$$\begin{aligned} {}^{\Gamma}/_{\Gamma_x} &\cong_\Gamma \Gamma\{y\} \\ \gamma &\mapsto \gamma(y) \end{aligned}$$

So counting orbit points is same as counting:

> **Definition.** Orbit counting function
>
> > Let
> > $$\Gamma \leq O^+(1,n)(\mathbb{R}) \cong \mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n) \curvearrowright \mathbb{R}\boldsymbol{H}^n$$
> > be a discrete subgroup.
>
> Let $x, y \in \mathbb{R}\boldsymbol{H}^n$ and $r > 0$. Then
> $$\begin{aligned} N(r,x,y) &:= |\{\gamma \in \Gamma | d(x, \gamma(y)) < r\}| \\ &= |\{\gamma \in \Gamma | \gamma(y) \in B_r(x)\}| \end{aligned}$$
> is the **orbit counter** that counts *how many points of $\Gamma\{y\}$ is in the ball $B_r(x)$*.



- Let $\gamma_0 \in \Gamma$.
    - Then
    $$N(r,x,y) = N(r,x,\gamma_0 y)$$
    - There is a $r > 0$ small enough such that
    $$N(r,x,y) = \begin{cases} 1 \\ |\Gamma_x| \end{cases}$$

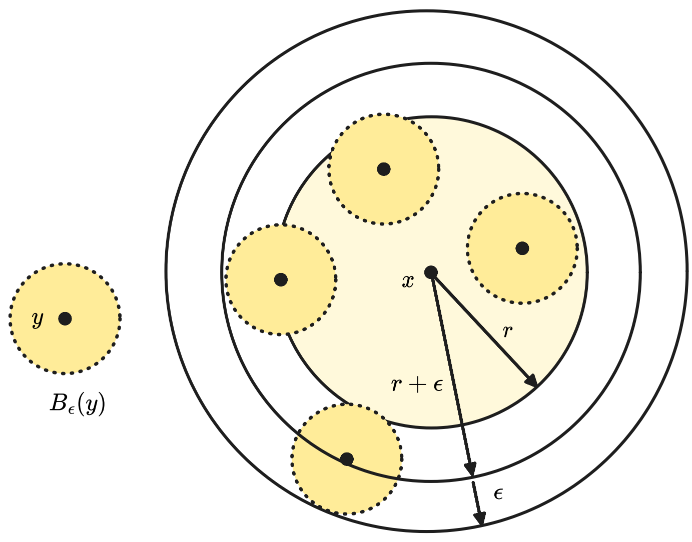
the orbit count is bounded above by the exponential in r

> > Let
> > $$\Gamma \leq O^+(1,n)(\mathbb{R}) \cong \mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n) \curvearrowright \mathbb{R}\boldsymbol{H}^n$$
> > be a discrete subgroup.
>
> Let $y \in \mathbb{R}\boldsymbol{H}^n$. Then there is a $A_{\Gamma,y} > 0$ such that
> $$\forall x \in \mathbb{R}\boldsymbol{H}^n, N(r,x,y) \leq A_{\Gamma,y} e^{r(n-1)}$$

By the proper action of the discrete group $\Gamma \curvearrowright \mathbb{R}\boldsymbol{H}^n$, choose $\epsilon > 0$ be small enough so that

$$\{\gamma(B_\epsilon(y)) | \gamma \in \Gamma\}$$

do not overlap unless $\gamma \in \Gamma_y$.



- Then by the volume of balls

$$\begin{aligned}
\mu_{\mathbb{R}\boldsymbol{H}^n}(B_s(o)) &= m_{S^{n-1}}(S^{n-1})\int_{t\in(0,s)}(\sinh t)^{n-1}\mathrm{d}t \\
&\le \frac{m_{S^{n-1}}(S^{n-1})}{2^{n-1}}\int_{t\in(0,s)} e^{(n-1)t}\mathrm{d}t \\
&\le \frac{m_{S^{n-1}}(S^{n-1})}{2^{n-1}}\left(\frac{e^{(n-1)s}}{(n-1)}-1\right)
\end{aligned}$$

  we have

$$\begin{aligned}
\bigsqcup_{\gamma\in\Gamma:\gamma(y)\in B_r(x)}\gamma(B_\epsilon(y)) &\subset B_{r+\epsilon}(x) \\
\mu_{\mathbb{R}\boldsymbol{H}^n}(B_\epsilon)N(r,x,y) &\le \mu_{\mathbb{R}\boldsymbol{H}^n}(B_{r+\epsilon}(x)) \\
N(r,x,y) &\le \frac{\mu_{\mathbb{R}\boldsymbol{H}^n}(B_{r+\epsilon})}{\mu_{\mathbb{R}\boldsymbol{H}^n}(B_\epsilon)} \\
&= c_0\left(\frac{e^{(n-1)(r+\epsilon)}}{(n-1)}-1\right) \\
&\le c_1 e^{(n-1)r}
\end{aligned}$$

for lattices, the orbit count is bounded below by exponential in r *eventually*

Let Γ be a discrete subgroup of finite covolume, $o\in\mathbb{R}\boldsymbol{H}^n$ be such that $\Gamma_o=\{\mathrm{Id}\}$ and let

$$\begin{aligned}
D &:= \operatorname{int}(D(\Gamma,o)) \\
D(r) &:= B_r(o)\cap D
\end{aligned}$$

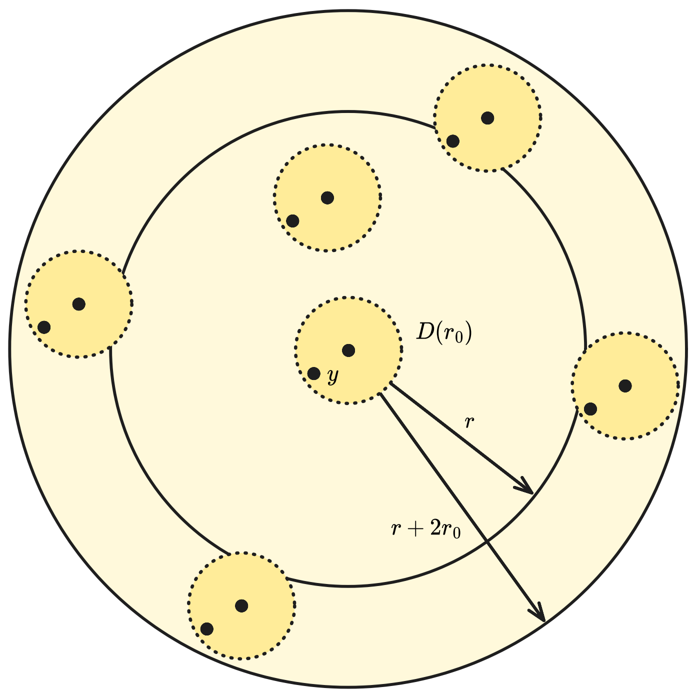
- Then

$$\begin{aligned}
B_r(o) &= B_r(o)\cap\Gamma\overline{D} \\
&= B_r(o)\cap\Gamma(D\sqcup\partial D) \\
&= B_r(o)\cap\Gamma(D(r_0)\sqcup(D\setminus D(r_0))\sqcup\partial D)
\end{aligned}$$

  for any $r_0>0$.
- Thus

$$\begin{aligned}
\mu(B_r) &= \mu(B_r\cap\Gamma D(r_0)) \\
&\quad+\mu(B_r\cap\Gamma(D\setminus D(r_0))) \\
&\quad+\cancelto{0}{\mu(B_r\cap\Gamma(\partial D))}
\end{aligned}$$

- $$\begin{aligned}
B_r(o)\cap\Gamma D(r_0) &= B_r(o)\cap\left(\bigsqcup_{\gamma\in\Gamma}\gamma(D(r_0))\right) \\
&\subseteq \bigsqcup_{\gamma\in\Gamma,d(\gamma(y),o)<r+2r_0}\gamma(D(r_0))
\end{aligned}$$

  for any fixed $y\in D(r_0)$.
- 

- Thus for any $y\in D(r_0)$ we have

$$\begin{aligned}
\mu(D(r_0))N(r+2r_0,o,y) &= \mu\left(\bigsqcup_{\gamma\in\Gamma,d(\gamma(y),o)<r+2r_0}\gamma(D(r_0))\right) \\
&\ge \mu\left(B_r(o)\cap\Gamma D(r_0)\right) \\
&= \mu(B_r)-\mu(B_r\cap\Gamma(D\setminus D(r_0))) \\
&= a_1\int_{t\in(0,r)}(\sinh t)^{n-1}\mathrm{d}t-\int_{y\in D\setminus D(r_0)}N(r,o,y)\mathrm{d}\mu \\
&\ge a_1\int_{t\in(0,r)}(\sinh t)^{n-1}\mathrm{d}t-a_2\epsilon e^{(n-1)r}
\end{aligned}$$

- For $R_0>0$

$$\begin{aligned}
&a_1\int_{t\in(0,r)}(\sinh t)^{n-1}\mathrm{d}t-a_2\epsilon e^{(n-1)r} \\
&= a_1\int_{t\in(0,R_0)}(\sinh t)^{n-1}\mathrm{d}t+a_1\int_{r\in(R_0,r)}(\sinh t)^{n-1}\mathrm{d}t-a_2\epsilon e^{(n-1)r} \\
&= a_3+a_1\int_{r\in(R_0,r)}\left(\frac{e^t-e^{-t}}{2}\right)^{n-1}\mathrm{d}t-a_2\epsilon e^{(n-1)r}
\end{aligned}$$

- For

$$\begin{aligned} \sinh t &\geq \frac{e^t}{4} \\ \iff 2e^t - 2e^{-t} &\geq e^t \\ \iff e^t - 2e^{-t} &\geq 0 \\ \iff e^{2t} &\geq 2 \\ \iff t &> R_0 \end{aligned}$$

for some $R_0 > 0$.
- Then

$$\int_{r\in(R_0,r)} \left(\frac{e^t - e^{-t}}{2}\right)^{n-1} \mathrm{d}t \geq \int_{r\in(R_0,r)} \frac{e^{(n-1)t}}{4^{n-1}} \mathrm{d}t$$

- Therefore

$$\begin{aligned} \mu(D(r_0))N(r+2r_0,o,y) &\geq a_3 + a_4(e^{(n-1)r} - e^{(n-1)R_0}) - a_2\epsilon e^{(n-1)}r \\ &\geq (a_4 - \epsilon a_2)e^{(n-1)r} + a_5 \end{aligned}$$

- So we must bound $(c_1 - \epsilon c_2)e^{\alpha r} + c_3$ by $c_4 e^{\alpha r}$ for some $c_4 > 0$ were of course we have $\epsilon > 0$ small enough so that $c_1 - \epsilon c_2 > 0$.
  - If $c_3 > 0$ then

  $$c_3 > 0 \implies (c_1 - \epsilon c_2)e^{\alpha r} + c_3 > (c_1 - \epsilon c_2)e^{\alpha r}$$

  - Let $c_3 < 0$. Consider

  $$\lim_{r\to\infty} \frac{(c_1 - \epsilon c_2)e^{\alpha r} + c_3}{e^{\alpha r}} = c_1 - \epsilon c_2 > 0$$

    - So there is some $R_1 > 0$ and $c_4 > 0$ such that

    $$\text{for } r > R_1, (c_1 - \epsilon c_2)e^{\alpha r} + c_3 \geq c_4 e^{\alpha r}$$

- Thus for $r > R$

$$N(r,o,y) \geq A' e^{(n-1)r}$$

Current note has 0 direct children and 0 total descendants.

- stem
  - O+ R1 n
    - Discrete subgroups of $O^+(1,n)$ $\leftrightarrow$ hyperbolic n-manifolds/orbifolds
      - Counting orbits of discrete subgroups of $O^+(1,n)(\mathbb{R})$ in $\mathbb{R}\boldsymbol{H}^n$

And it has 11 siblings.

- stem
  - O+ R1 n
    - Discrete subgroups of $O^+(1,n)$ $\leftrightarrow$ hyperbolic n-manifolds/orbifolds
      - Convex cocompact discrete subgroups
      - Combination of discrete subgroups of $O^+(1,n)(\mathbb{R})$
      - Exponential sum of a discrete subgroup of $\mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n)$
      - $\mathrm{RepDF}(F_g, O^+(1,n)(\mathbb{R}))$
      - Measure homology on a hyperbolic manifold
      - Discrete subgroups with limit set a discontinuum
      - Limit sets of discrete subgroups
      - Discrete subgroups with limit set a $k-1$-sphere
      - Counting orbits of discrete subgroups of $O^+(1,n)(\mathbb{R})$ in $\mathbb{R}\boldsymbol{H}^n$
      - Rigidity of $\mathbb{R}$-hyperbolic manifolds
      - $\mathrm{RepDF}(\pi_1(T_g^2, \bullet), O^+(1,n)(\mathbb{R}))$