Info

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Disk in holomorphic image of disk

(Bloch's theorem) Let $f \in \mathscr{O}(D)$ such that $f'(0) = 1$. Then there is a disk $B_r(a) \subseteq D$ on which f is injective and

$$B_\beta(f(a)) \subseteq f(B_r(a))$$

where $\beta \geq \frac{3}{2} - \sqrt{2} > \frac{1}{12}$.

a bounded $f \in \mathscr{O}(D)$ contains a disk centered at $f(0)$ of radius $\frac{|f'(0)|}{6\|f - f(0)\|_\infty}$

> **Proposition:** Let $f \in \mathscr{O}^\infty(D)$ such that $f(0) = 0$, $f'(0) = 1$. Then $\|f\|_\infty \geq 1$ and
>
> $$B\left(\frac{1}{6\|f\|_\infty}, 0\right) \subseteq f(D)$$

Let $r \in (0, 1)$ and

$$f(z) = z + a_2 z^2 + \dots$$

where by

- By the Cauchy integral formula,

$$\begin{aligned}\frac{\left|f^{(n)}(0)\right|}{n!} &\leq \frac{1}{2\pi} \frac{\|f|_{rS^1}\|_\infty}{r^{n+1}} (2\pi r) \\ &= \frac{\|f|_{rS^1}\|_\infty}{r^n}\end{aligned}$$

we have for each $n \geq 1$

$$\begin{aligned}\forall r \in (0,1), |a_n| &\leq \frac{\|f\|_\infty}{r^n} \\ \implies |a_n| &\leq \|f\|_\infty\end{aligned}$$

- In particular, for $n = 1$ we have $1 = |a_1| \leq \|f\|_\infty$
- Then

$$\begin{aligned}f(z) - \sum_{n\geq 2} a_n z^n &= z \\ |f(z)| + \left|\sum_{n \geq 2} a_n z^n\right| &\geq |z| \\ |f(z)| &\geq |z| - \sum_{n\geq 2} |a_n z^n|\end{aligned}$$

- For $|z| = \frac{1}{4\|f\|_\infty}$ we have

$$|f(z)| \geq \frac{1}{4\|f\|_\infty} - \sum_{n\geq 2} \|f\|_\infty \frac{1}{(4\|f\|_\infty)^n} \geq \frac{1}{6\|f\|_\infty}$$

- Thus

$$\partial B\left(\frac{1}{4\|f\|_\infty}, 0\right) \overset{f}{\mapsto} B^{c}\left(\frac{1}{6\|f\|_\infty}, 0\right)$$

- By [Rouche's theorem]

$$B\left(\frac{1}{6\|f\|_\infty}, 0\right) \subseteq f(D)$$

Analogously,

> **Proposition:** Let $f \in \mathscr{O}^\infty(B(R_0, z_0))$. Then
>
> $$B\left(\frac{R_0^2 |f'(0)|^2}{6\|f - f(z_0)\|_\infty}, f(z_0)\right) \subseteq f(B(R_0, z_0))$$

Consider

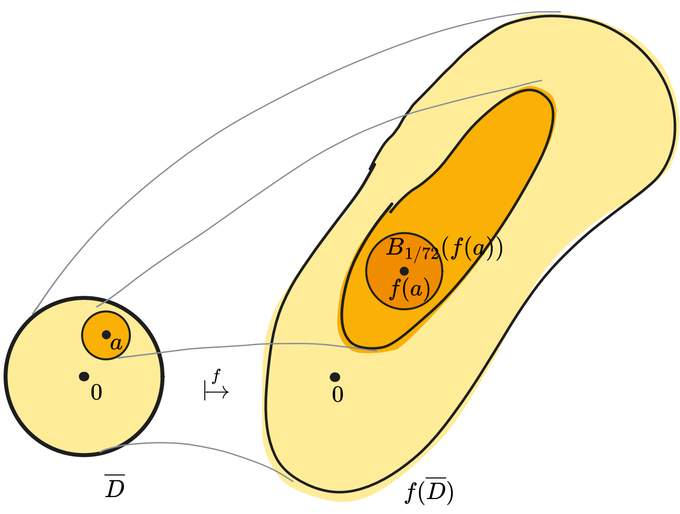
$$g(z) :\equiv \frac{1}{R_0 g'(0)} f(R_0 z)?$$

and apply

> **Proposition:** Let $f \in \mathscr{O}^\infty(D)$ such that $f(0) = 0$, $f'(0) = 1$. Then $\|f\|_\infty \geq 1$ and
>
> $$B\left(\frac{1}{6\|f\|_\infty}, 0\right) \subseteq f(D)$$

a $f \in \mathscr{O}(\overline{D})$ contains an inveritble disk of radius $\frac{|f'(0)|}{72}$

> **Proposition:** Let $f \in \mathscr{O}(\overline{D})$ such that $f(0) = 0$ and $f'(0) = 1$. Then there is a disk $B(r, a) \subseteq D$ on which f is injective and
>
> $$B\left(\frac{1}{72}, f(a)\right) \subseteq f(B(r, a))$$



Let

$$
\begin{aligned}
h &: [0,1] \to \mathbb{R} \\
h(r) &:= (1-r)\left\| f'|_{\partial B(r,0)} \right\|_\infty
\end{aligned}
$$

- h is continuous, $h(0) = 1, h(1) = 0$
- **Definition.** $r_0 := \sup h^{-1}(1) \in (0,1)$
- Choose $a \in D$ such that $|a| = r_0$ and $|f'(a)| = \left\| f'|_{\partial B(r_0,0)} \right\|_\infty$.
- Then

$$
h(r_0) = 1 \implies |f'(a)| = \frac{1}{1-r_0}
$$

- For $|z-a| < \frac{1}{2}(1-r_0) =: \rho_0$ we have $|z| < \frac{1}{2}(1+r_0)$. Then for $|z-a| < \rho_0$ we have

$$
\begin{aligned}
|f'(z) - f'(a)| &\le |f'(z)| + |f'(a)| \\
&\le \left\| f'|_{\partial B(\frac{1}{2}(1+r_0),0)} \right\| + \frac{1}{1-r_0} \\
&= \underbrace{h\left(\underbrace{\frac{1+r_0}{2}}_{<r_0}\right)}_{<1} \frac{1}{1-\frac{1+r_0}{2}} + \frac{1}{1-r_0} \\
&< \frac{1}{\rho_0} + \frac{1}{2}\rho_0 \\
&= \frac{3}{2\rho_0}
\end{aligned}
$$

- By

> **Proposition:** Consider a holomorphic function f such that
>
> $$f : B(R_0, z_0) \to B(M_0, f(z_0))$$
>
> that is
>
> $$|f(z) - f(z_0)| < M_0 \qquad |z - z_0| < R_0$$
>
> Then
>
> $$|f(z) - f(z_0)| \le \frac{M_0}{R_0}|z - z_0| \qquad |z - z_0| < R_0$$

we have

$$
|f'(z) - f'(a)| \le \frac{3}{2\rho_0}\frac{1}{\rho_0}|z - z_0| \qquad \text{for } |z-a| < \rho_0
$$

- Then for $r := \frac{1}{3}\rho_0$ we have

$$
\begin{aligned}
|f'(z) - f'(a)| &\le \frac{3}{2\rho_0}\frac{1}{\rho_0}\frac{1}{3}\rho_0 \\
&= \frac{1}{2\rho_0} \qquad \text{for } |z-a| < r \\
&= |f'(a)|
\end{aligned}
$$

- By

1.3 Lemma. *Let f be an analytic function on the disk $B(a;\ r)$ such that $|f'(z) - f'(a)| < |f'(a)|$ for all z in $B(a; r)$, $z \neq a$; then f is one-one.*

Proof. Suppose z_1 and z_2 are points in $B(a;\ r)$ and $z_1 \neq z_2$. If γ is the line segment $[z_1, z_2]$ then an application of the triangle inequality yields

$$
\begin{aligned}
|f(z_1) - f(z_2)| &= \left| \int_\gamma f'(z)\, dz \right| \\
&\ge \left| \int_\gamma f'(a)\, dz \right| - \left| \int_\gamma [f'(z) - f'(a)]\, dz \right| \\
&\ge |f'(a)|\, |z_1 - z_2| - \int_\gamma |f'(z) - f'(a)|\, |dz|.
\end{aligned}
$$

Using the hypothesis, this gives $|f(z_1) - f(z_2)| > 0$ so that $f(z_1) \neq f(z_2)$ and f is one-one. ■

we have f is injective on $B_r(a)$.

- As #Wiki/inprogress

$$
|f(z) - f(a)| < \frac{1}{3} \qquad \text{for } |z-a| < \frac{1}{3}\rho_0
$$

from

> **Proposition:** Let $f \in \mathscr{O}^\infty(B(R_0, z_0))$. Then
>
> $$B\left(\frac{R_0^2 |f'(0)|^2}{6\|f - f(z_0)\|_\infty}, f(z_0)\right) \subseteq f(B(R_0, z_0))$$

> we conclude
>
> $$B\left(\underbrace{\frac{\left(\frac{1}{3}\rho_0\right)^2\left(\frac{1}{2\rho_0}\right)^2}{6\frac{1}{3}}}_{\frac{1}{9\cdot 4\cdot 2}=\frac{1}{72}},0\right)\subseteq f\left(B\left(r,a\right)\right)$$

Consider the derivative

$$\mathfrak{D}_0:\mathscr{O}((\overline{D},0),(\mathbb{C},0))\to\overline{D}$$

Then the subset of functions such that $f'(0)=1$ is

$$\mathscr{O}((\overline{D},0),(\mathbb{C},0))\cap\mathfrak{D}_0^{-1}(1)$$

Now,

$$\mathscr{O}((\overline{D},0),(\mathbb{C},0))\cap\mathfrak{D}_0^{-1}(1)\to\left[\frac{1}{72},\infty\right)$$
$$f\mapsto\sup\left\{r\geq 0\middle|f^{-1}(B(r,b)\xhookrightarrow{f}B(r,b)\right\}$$

> **Proposition:** Let $U\subseteq\mathbb{C}$ be a bounded open set and $f\in\mathscr{O}(U)\cap\mathcal{C}(\overline{U})$. Consider
>
> $$s_0:=\min_{z\in\partial U}|f(z)-f(a)|$$
>
> for some $a\in U$. Then
>
> $$B(s,f(a))\subseteq f(U)$$

As $f(\partial U)\subseteq\mathbb{C}$ is compact and does not include $f(a)$ there is a minimum $s_0>0$ for

$$|f(z)-f(a)| \qquad z\in\partial U$$

> - As $f(U)$ is open

Current note has 0 direct children and 0 total descendants.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients ${}_\Gamma\backslash^{\mathbb{R}^n}, {}_\Gamma\backslash^{\mathbb{C}^n}$
    - Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - mapping
        - Disk in holomorphic image of disk

And it has 13 siblings.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients ${}_\Gamma\backslash^{\mathbb{R}^n}, {}_\Gamma\backslash^{\mathbb{C}^n}$
    - Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - mapping
        - Bounding derivative of holomorphic mappings
        - Bounding holomorphic functions linearly
        - Branch points and ramification index of holomorphic functions
        - Disk in holomorphic image of disk
        - Size of holomorphic image of disks
        - Maximum and minimum of holomorphic functions
        - Fibers of holomorphic mappings
        - Non-constant holomorphic functions locally look like w^k and are open mappings
        - Biholomorphic mapping onto disk
        - Conformal radius of pointed simply connected open subsets of $\mathbb{C}$
        - Holomorphic functions on punchered disk
        - Holomorphic function whose singularities are not isolated
        - A holomorphic map branched over $\mathbb{Q}$