Info

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Probability distribution of flow

[1]

$$\Phi_F^t(x) = x + tF(x) + O(t^2)$$

$$\det \mathfrak{D}_x \Phi_F^t = 1 + t(\mathrm{div} F(x)) + O(t^2)$$

$F \in \mathrm{Vec}^1(\mathbb{R}^n)$ be a complete vector field, then

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{vol}_{\mathbb{R}^n}(\Phi_F^t A)\Big|_{t=0} = \int_A \mathrm{div} F$$

$F \in \mathrm{Vec}^1(\mathbb{R}^n)$ be a complete vector field, $\mathrm{div} F = 0 \implies \Phi_F^t$ preserves volume,

$$\mathrm{vol}_{\mathbb{R}^n}(\Phi_F^t A) = \mathrm{vol}_{\mathbb{R}^n}(A)$$

Definition. Evolution of a probability distribution by flows of a vector field

Let ρ_0 be a probability distribution on $\mathbb{R}^n$ then define the distribution ρ_t^F evolved by F as that satisfies

$$\int_A \rho_t^F = \int_{\Phi_F^{-t} A} \rho_0$$

for any $A \subset \mathbb{R}^n$.

- $$\int_{y \in A} \rho_t^F(y) = \int_{x \in \Phi_F^{-t} A} \rho_t^F(\Phi_F^t(x))(\det \mathfrak{D}_x \Phi_F^t) = \int_{x \in \Phi_F^{-t} A} \rho_0(x)$$
  - As A is arbitrary, the integrands must be equal
    $$\iff \rho_t^F(\Phi_F^t(x))(\underbrace{\det \mathfrak{D}_x \Phi_F^t}_{1 + t(\mathrm{div} F(x)) + O(t^2)}) = \rho_0(x)$$
    - $$\implies \rho_0 = \rho_t(\Phi^t)(1 + t\mathrm{div} F + O(t^2))$$
      - $$\iff \rho_0 - \rho_t(\Phi^t) = \rho_t(\Phi^t)(t\mathrm{div} F + O(t^2))$$

$F \in \mathrm{Vec}^1(\mathbb{R}^n)$ be a complete vector field,

$$\frac{\mathrm{d}}{\mathrm{d}t} \rho_t \circ \Phi_F^t = \partial_t \rho_t + \partial_F(\rho_t) = -\rho_t \mathrm{div} F$$

Current note has 0 direct children and 0 total descendants.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, _\Gamma\backslash^{\mathbb{C}^n}$
    - ODEs in $\mathbb{R}^n$ $\leftrightarrow$ Vector fields in $\mathbb{R}^n$
      - Probability distribution of flow

And it has 10 siblings.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, _\Gamma\backslash^{\mathbb{C}^n}$
    - ODEs in $\mathbb{R}^n$ $\leftrightarrow$ Vector fields in $\mathbb{R}^n$
      - Constant of a flow in $\mathbb{R}^n$
      - Euler's iterative solution method for first order ODEs
      - Existence of integral curves of vector fields in $\mathbb{R}^n$
      - Fixed points
      - Flow of a vector field in $\mathbb{R}^n$
      - Graph $\to$ polynomial ODE
      - Gradient flows on $\mathbb{R}^n$
      - Hamiltonian vector fields in $\mathbb{R}^{2n}$ with standard symplectic form
      - Probability distribution of flow
      - Volume change by flows on $\mathbb{R}^n$

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1. Roderich Tumulka - Lecture Notes on Mathematical Statistical Physics.pdf > page=36 ↩