Info

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Density on smooth manifolds

Definition. Density on a $\mathbb{R}$-vector space

A **density** on a finite dimensional $\mathbb{R}$-vector space V is a function

$$\mu : V^{\dim V} \to \mathbb{R}$$

such that

$$T \in \mathrm{EndVec}_{\mathbb{R}}(V),\ v_1, \dots, v_n \in V \implies \mu(Tv_1, \dots, Tv_n) = |\det T| \mu(v_1, \dots, v_n)$$

The space of all densities

$$\mathcal{D}(V)$$

is a $\mathbb{R}$-vector space under pointwise addition and scalar multiplication.

If $\mu_1, \mu_2 \in \mathcal{D}(V)$ and they agree on a basis of V, then $\mu_1 = \mu_2$.

We have a map

$$\begin{aligned} \Lambda^{\dim V}(V^*) &\to \mathcal{D}(V) \\ \omega &\mapsto |\omega| \end{aligned}$$

and for any $|\omega|$ we have an $\mathbb{R}$-linear isomorphism

$$\begin{aligned} \mathbb{R} &\to \mathcal{D}(V) \\ c &\mapsto c\,|\omega| \end{aligned}$$

and thus

$$\dim_{\mathbb{R}} \mathcal{D}(V) = 1$$

Definition. A **positive density** on V is a density μ satisfying $\mu > 0$ for some linearly independent tuple.

Definition. Density bundle on a smooth manifold

Let M be a smooth manifold with or without boundary. Then

$$\mathcal{D}M := \coprod_{p \in M} \mathcal{D}(T_p M)$$

with the projection

$$\begin{aligned} \pi : \mathcal{D}M &\to M \\ \mathcal{D}(T_p M) &\mapsto p \end{aligned}$$

has a natural smooth line bundle structure over M called the **density bundle**.
A section of $\mathcal{D}(M)$ is called a **density** on M.

$$\begin{aligned} \Omega^{\dim M}(M) &\to \mathcal{D}M \\ \omega &\mapsto |\omega| \end{aligned}$$

is a bijection?

A smooth positive density exists on every smooth manifold with or without boundary.

Definition. Pullback of densities

If

$$F : M \to N$$

is a smooth map between smooth n-manifolds with or without boundary and μ is a density on N, we define **pullback density** on M by

$$(F^*\mu)_p(v_1, \dots, v_n) := \mu_{F(p)}(F_p^* v_1, \dots, F_p^* v_n)$$

This gives a

$$\begin{aligned} F^* : \mathcal{D}N &\to \mathcal{D}M \\ \mu &\mapsto F^*\mu \end{aligned}$$

Definition. Integration of densities on smooth manifolds

Let M be a smooth manifold with or without boundary.

- If μ is a density on M supported on a domain of a single smooth chart

  $$x : U \subseteq M \to \mathbb{R}^{\dim M}$$

  then the **integral of** μ **on** M is defined as

  $$\int_M \mu = \int_{x(U)} (x^{-1})^* \mu$$

- This is extended to arbitrary densities $\mu \in \mathcal{D}M$ by setting

$$\int_M \mu = \sum_i \int_M \psi_i \mu$$

where $\{\psi_i\}$ is a smooth partition of unity subordinate to a open cover of M by smooth charts.

Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - manifolds
    - smooth $\mathbb{R}$-manifold
      - Density on smooth manifolds

And it has 19 siblings.

- structures based on sets
  - manifolds
    - smooth $\mathbb{R}$-manifold
      - Smooth Lie group action on a smooth manifold
      - Borel measures on a smooth manifold
      - G-bundles over smooth manifolds $G \curvearrowright P \to X$
      - Vector bundle on smooth manifolds
      - Curves in smooth manifolds
      - Density on smooth manifolds
      - Functions between smooth manifolds
      - Functions from smooth manifolds to $\mathbb{R}$
      - Flows on a smooth manifold
      - Locally $G \curvearrowright X$-manifolds
      - Smooth singular homology of a smooth manifold
      - Isomorphisms and automorphisms of smooth manifolds
      - Lagrangian flows
      - Local transformation rules
      - Moduli space of Riemannian metrics on a smooth manifold
      - Moduli space of quotients
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      - Tangents on a smooth manifold