Info

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Discrete subgroup of isometry group of a simply connected Riemannian manifold of negative curvature

Definition. Limit set and exponent of a discrete subgroup of isometry group of a simply connected Riemannian manifold of negative curvature

Let X be a simply connected Riemannian manifold of negative curvature with visual boundary at infinity ∂X, compactification $\overline{X}$ and $\Gamma \leq \mathrm{Isom}(X)$ be a discrete subgroup. Then for some $x \in X$

- the **limit set** of Γ is

$$\Lambda(\Gamma) := \overline{\Gamma\{x\}} \cap \partial X \subseteq \overline{X}$$

- the **exponent** of Γ is

$$\begin{aligned}\delta_\Gamma &:= \inf\left\{s \in \mathbb{R} \middle| \sum_{\gamma\in\Gamma} \exp(-sd_X(x,\gamma x))\right\}\\ &= \limsup_{r\to\infty} \frac{1}{r}\log|\{\gamma\in\Gamma | d(x,\gamma x) \leq r\}|\end{aligned}$$

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Assume the volume entropy of X is finite $\delta_X < \infty$. Then

$$\delta_\Gamma \leq \delta_X$$

If Γ is cocompact then $\delta_\Gamma = \delta_X$ and $\Lambda(\Gamma) = \partial X$.

As Γ is discrete, there exists $s > 0$ such that for all $\gamma \in \Gamma$

$$B_s(\gamma x) \cap B_s(x) \neq \emptyset \implies \gamma \in \Gamma_x$$

- For $r > 0$ we have

$$|\{\gamma\in\Gamma | d(x,\gamma x) \leq r\}| \mu(B_s(x)) \leq |\Gamma_x| B_{r+s}(x)$$

- Let Γ be a cocompact discrete subgroup of $\mathrm{Isom}(X)$.
  - There exists $s > 0$ such that

$$X = \bigcup_{\gamma\in\Gamma} B(\gamma x, s)$$

  - Thus for $r \geq 0$ we have

$$B_r(x) \subset \bigcup_{\substack{\gamma\in\Gamma\\ d(x,\gamma x)\leq r+s}} B_s(\gamma x)$$

Let Γ be a non-elementary discrete group of isometries of X. Then there exists a Γ-conformal density of dimension δ_Γ with support $\Lambda(\Gamma)$.

Let $x \in X$ and suppose $\sum_{\gamma\in\Gamma} \exp(-\delta_\Gamma d(x,\gamma x)) = \infty$.

- Then for

$$\begin{aligned}\Phi : (\delta_\Gamma, \infty) &\to X\\ s &\mapsto \sum_{\gamma\in\Gamma} \exp(-sd(x,\gamma x))\end{aligned}$$

  we have

$$\nu_s := \frac{1}{\Phi(s)} \sum_{\gamma\in\Gamma} \exp(-sd(x,\gamma x))\delta_{\gamma x}$$

  a probability measure on $\overline{X}$.

- For a sequence $s_n \xrightarrow{n\to\infty} \delta_\Gamma$ we have a sequence

$$\nu_{s_{n_k}} \xrightarrow{k\to\infty} \nu$$

  weakly

- ...

(Sullivan's preservation of cross ratios) Let X_1, X_2 be two Riemannian manifolds of sectional curvature ≤ -1 and let

$$\varphi : \Gamma_1 \to \Gamma_2$$

with $\delta(\Gamma_1) = \delta(\Gamma_2) = \delta$ is a group homomorphism and

$$\Phi : (\Lambda(\Gamma_1), \nu_1) \to (\Lambda(\Gamma_2), \nu_2)$$

is a Borel bi-measurable non-singular map φ-conjugating the actions

$$\forall \gamma \in \Gamma_1, \Phi \circ \gamma = \phi(\gamma) \circ \Phi$$

If

$$(\Phi \times \Phi)^* \mu_2 = c\mu_1$$

then Φ preserves cross-ratios ν_1-almost everywhere on $\Lambda(\Gamma_1)$.

Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - manifolds
    - $\mathbb{R}$-manifold semi-Riemannian
      - Complete Riemannian manifolds
        - Riemannian manifolds with negative curvature
          - Discrete subgroup of isometry group of a simply connected Riemannian manifold of negative curvature

And it has 4 siblings.

- structures based on sets
  - manifolds
    - $\mathbb{R}$-manifold semi-Riemannian
      - Complete Riemannian manifolds
        - Riemannian manifolds with negative curvature
          - Geodesic flow on Riemannian manifolds of negative curvature
          - Discrete subgroup of isometry group of a simply connected Riemannian manifold of negative curvature
          - Rigidity of marked length spectrum for Riemannian 2-manifolds of negative curvature
          - Equivalence of marked length spectrum, geodesic flow and boundary of universal cover of Riemannian manifolds with negative curvature