Info

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Cartan geometry

Definition. Definition

A **model for a Cartan geometry** consists of

- a faithful Klein pair, that is a pair of Lie algebras
$$\mathfrak{h} \leq \mathfrak{g}$$
such that largest (hence, any) ideal $\mathfrak{k} \trianglelefteq \mathfrak{g}$ contained in $\mathfrak{h}$ is trivial $\mathfrak{k} = 0$
- a Lie group H with Lie algebra $\mathfrak{h}$
- a representation
$$\rho : H \to \mathrm{AutLieAlg}_{\mathbb{R}}(\mathfrak{g})$$
extending $\mathrm{Ad}_H : H \to \mathrm{AutLieAlg}_{\mathbb{R}}(\mathfrak{h})$.

The kernel of the representation ρ is called **kernel** of the model geometry.
If the kernel is trivial, then the model geometry is called **faithful**.
If $\mathfrak{h}$ is a maximal subalgebra of $\mathfrak{g}$ then the model geometry is called **primitive**.
If there is an Ad_H-module decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ then the model geometry is called **reductive**.

The Lie algebra of $\ker \rho \trianglelefteq H$ is an ideal? of $\mathfrak{g}$ contained in $\mathfrak{h}$, so $\ker \rho$ must be discrete.

from Cartan gauges to a principal H-bundle

Let $(\mathfrak{h} \leq \mathfrak{g}, \rho : H \to \mathrm{AutLieAlg}_{\mathbb{R}}(\mathfrak{g}))$ be a *model Cartan geometry*. A **Cartan gauge** with this model on a smooth manifold M is an open set $U \subseteq M$ and a $\mathfrak{g}$-valued 1-form on U
$$\theta_U : TU \to \mathfrak{g}$$
such that
$$T_pU \xrightarrow{\theta_U} \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$$
is a linear isomorphism for each $p \in U$.

Cartan geometry

Definition. Cartan geometry

A **Cartan geometry** (P, ω) on a smooth manifold M modelled on the *model Cartan geometry* $(\mathfrak{h} \leq \mathfrak{g}, \rho : H \to \mathrm{AutLieAlg}_{\mathbb{R}}(\mathfrak{g}))$ is a principal right H-bundle
$$\begin{tikzcd} P \arrow[d] & H \arrow[l, bend right] \\ M & \end{tikzcd}$$
and a $\mathfrak{g}$-valued 1-form ω on P satisfying

- $\omega_p : T_pP \to \mathfrak{g}$ is an $\mathbb{R}$-isomorphism
- $R_h^* \omega = \rho_{h^{-1}} \omega$ for all $h \in H$
- $\omega(X) = X$ for all $X \in \mathfrak{h}$

The **Cartan curvature** of the Cartan geometry is the $\mathfrak{g}$ valued 2-form
$$\Omega := \mathrm{d}\Omega + \frac{1}{2}[\omega, \omega]$$
The image of
$$\Lambda^2 TP \xrightarrow{\Omega} \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$$
defined to be the **torsion**.
The geometry is **complete** if every ω-constant vector fields on P are complete.
A Cartan geometry is **faithful**, **primitive** or **reductive** if the model geometry is faithful, primitive or reductive.

flat Cartan geometries
