Info

This note, that was found here, as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
was created on June 6, 2025 4:18:22 PM,
was last modified on September 7, 2026 2:34:28 PM, and
was exported on September 7, 2026 3:04:06 PM.

Left translation $S^1 \curvearrowright S^1$, *AKA* rotations on the circle

Definition. Left translation $S^1 \curvearrowright S^1$, *AKA* rotations on the circle

The left translation of the circle group (by the element $\exp(i2\pi\theta) \in S^1$) S^1

$$\begin{aligned} S^1 &\to S^1 \\ z &\mapsto \exp(i2\pi\theta)z \end{aligned}$$

for $\theta \in \mathbb{R}$ is also called a **rotation** on the circle.

If $\theta \in \mathbb{Q}$ the transformation is called a **rational rotation**, that is, the action by $\mathbb{Q}/\mathbb{Z} \subset \mathbb{R}/\mathbb{Z} \cong S^1$.
If $\theta \in \mathbb{R} \setminus \mathbb{Q}$, then its a **irrational rotation**.

rational rotations are periodic

Consider the *rational* rotation

$$R^{\frac{p}{q}}(z) := \exp\left(2\pi i \frac{p}{q}\right) z$$

where $p, q \in \mathbb{Z}_{>0}$. After q many iterations we have

$$\left(R^{\frac{p}{q}}\right)^q (z) = \exp(2\pi i p) z = z$$

Thus every point in S^1 is periodic under iterations of $R^{\frac{p}{q}}$.

For any rotation by $\exp(i\theta)$ if

$$\exp(i\theta)^k = 1 \implies k\theta \in 2\pi\mathbb{Z} \implies \frac{\theta}{2\pi} \in \mathbb{Q}$$

irrational rotations are minimal

Consider the rotation

$$R^\theta(z) := \exp(2\pi i\theta) z$$

where $\theta \in \mathbb{R} \setminus \mathbb{Q}$ (irrational).

- If there is any $z \in S^1$ with $k > 1$ such that

$$\begin{aligned} & (R^\theta)^k(z) &= z \\ \implies & \exp(2\pi i k\theta) z &= z \\ \implies & \exp(2\pi i k\theta) &= 1 \\ \implies & k\theta &\in \mathbb{Z} \\ \implies & \theta &\in \mathbb{Q} \end{aligned}$$

  which is a contradiction to $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Therefore, there are no periodic orbits of irrational rotation R^θ.

  > - Points are not periodic $\iff$ orbit A_θ is infinite as rotations are bijective.

> **Proposition:** Consider the forward orbit of $1 \in S^1$, that is,
>
> $$A_\theta := \left\{ (R^\theta)^k(1) = \exp(2\pi i\theta k) \mid k \in \mathbb{N} \right\}$$
>
> and any other forward orbit is a translate of A_θ
>
> $$\{ (R^\theta)^k(z) = \exp(2\pi i\theta k) z \mid k \in \mathbb{N} \} = A_\theta z$$
>
> The set
>
> $$A_\theta \subset S^1$$
>
> is **dense** (and therefore $A_\theta z$ is also dense in S^1).
> Therefore, every orbit of an irrational rotation is dense in the circle making irrational rotations *minimal*.

proof by circle partitions

- *Assume* A_θ is not dense in the circle S^1, so there is an interval $I \subset S^1$ of size $\epsilon > 0$ which does not intersect A_θ.
  - *Assume* there are $m > n$ such that

    $$d(\exp(2\pi i\theta m), \exp(2\pi i\theta n)) = d < \epsilon$$

  - Then rotating by $(R^\theta)^{m-n}$

    $$d(\exp(2\pi i\theta(m-n))z, z) = d < \epsilon$$

  - So for any $z \in A_\theta$, we may partition $S^1 \setminus I$ into roughly $(1-\epsilon)/d$ intervals of size d (and one a little smaller) and put some finite iteration of z into I.
  - This contradicts I not intersecting A_θ, this there is no such m, n, implying points in A_θ at the least ϵ distance apart.
  - But, that would mean there are at most $1/\epsilon$ elements in A_θ which contradicts the orbit A_θ being infinite.

    > - Points are not periodic $\iff$ orbit A_θ is infinite as rotations are bijective.

Alternatively, we may construct a sub-orbit with consecutive distance $< \epsilon$:

- Consider an arbitrary integer $N > 0$ and the N arcs of length $\frac{1}{N}$ of $\mathbb{R}/\mathbb{Z}$ with boundary points

$$0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}$$

  - Since the orbit A_θ is infinite there exists integers $n_1 < n_2$ such that

$$|(n_1 - n_2)\theta| \bmod 1 < \frac{1}{N}$$

  - Thus for every $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{Z}$ such that

$$n_\epsilon \theta \bmod 1 < \epsilon$$

  - Thus the distance between the consecutive entries

$$0, n_\epsilon\theta \bmod 1, 2n_\epsilon\theta \bmod 1, 3n_\epsilon\theta \bmod 1, \ldots$$

    is $< \epsilon$. Thus it must require finitely many (roughly $\frac{1}{\epsilon}$) entries from the front to partition $\mathbb{R}/\mathbb{Z}$ into size length $< \epsilon$.
  - Any $\alpha \in \mathbb{R}/\mathbb{Z}$ must be in one of those partitions, thus there is a k such that

$$|\alpha - kn_\epsilon\theta| < \epsilon$$

  - Thus A_θ is dense in $\mathbb{R}/\mathbb{Z}$.

Alternative, even simply we can argue about the complement of the closure of the orbit

- Let $T(z) = \lambda z$ be an irrational circle rotation. Consider an orbit $X \subset S^1$, and the complement of its closure

$$Y := S^1 \setminus \bar{X}$$

  - *Assume* Y is not empty. Because $\bar{X}$ is invariant and closed, Y is invariant and open.
  - Any open subset of S^1 is disjoint union of intervals. Consider the collection of disjoint intervals $\{L_1, \ldots, L_n\}$ of a fixed length. There must be finitely many of them as total length of S^1 is finite.
  - Because rotation is length preserving, $\bigcup_i L_i$ must be an invariant set, and because these sets are disjoint, the rotation must permute these sets.
  - So there is a $k > 0$ such that $T^k(L_1) = L_1$
    - that is the rotation T^k on $\mathbb{R}^2$ must preserve the proper arc $L_1 \subset S^1$
    - *OR* the endpoint of L_1 would map to itself, meaning its a fixed point of T^k, but the only rotation with a fixed point is the identity
  - Thus $T^k = \text{Id}$ making T periodic, which contradicts it being irrational.
  - Hence, Y is empty, making every orbit dense $\iff$ no proper closed non-empty subset.

proof by dense subgroups

- Consider

$$B_\theta := \{n\theta + m : n, m \in \mathbb{Z}\}$$

  then

$$B_\theta \xmapsto{\exp(2\pi i -)} A_\theta$$

  - Clearly, $(B_\theta, +)$ is a subgroup of $(\mathbb{R}, +)$.
  - If $0 \in B_\theta$ is a *limit point* of B_θ, then there is a $x \in B_\theta$ such that

$$0 < x < \epsilon$$

    - Then for any $y \in \mathbb{R}$, there exists $n \in \mathbb{Z}$ such that

$$nx \leq y < (n+1)x$$

      which shifts to

$$y - nx < x < \epsilon$$

      making nx

ergodicity

spectrum of circle rotations

We have the unitary map $R^{\theta *}$ on $L^2(S^1)$ such that

$$R^{\theta *}(f)(z) := f(R^\theta(x))$$

On the Fourier basis $(e_k)_{k\in\mathbb{Z}}$ this acts by

$$R^{\theta *}(e_k) = \exp(2\pi i\theta)^k e_k$$

so it is diagonal with eigenvalues A_θ.

- If θ is rational, then its spectral measure is

$$\mu = \sum_{x\in A_\theta} \delta_x$$

  with multiplicity $M\Big|_{A_\theta} \equiv \infty$.
- If θ is irrational then its spectral measure is

$$\mu = \sum_{x\in A_\theta} \delta_x$$

  with multiplicity $M\Big|_{A_\theta} \equiv 1$

An irrational rotation of the circle has *pure point spectrum of multiplicity 1* supported on $\langle\lambda\rangle < S^1$ for some $\lambda \in S^1$ of infinite order that is either the rotation angle or its inverse. Thus, two irrational rotations by $\lambda_1, \lambda_2 \in S^1$ are isomorphic $\iff \lambda_1 = \lambda_2^{\pm}$.

Current note has 0 direct children and 0 total descendants.

- stretchy
  - S^1
    - Left translation $S^1 \curvearrowright S^1$

And it has 4 siblings.

- stretchy
  - S^1
    - The map $z \mapsto z^n$ on the circle
    - Left translation $S^1 \curvearrowright S^1$
    - Homeomorphisms $S^1 \to S^1$
    - Loops in S^1 upto homotopy