Info

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Mean value property of functions

From

(Extremum value theorem for continuous $[a,b] \to \mathbb{R}$) Let

$$f : [a,b] \to \mathbb{R}$$

be **continuous**, then a global maxima *and* a minima exists $M, m \in [a,b]$ for f.

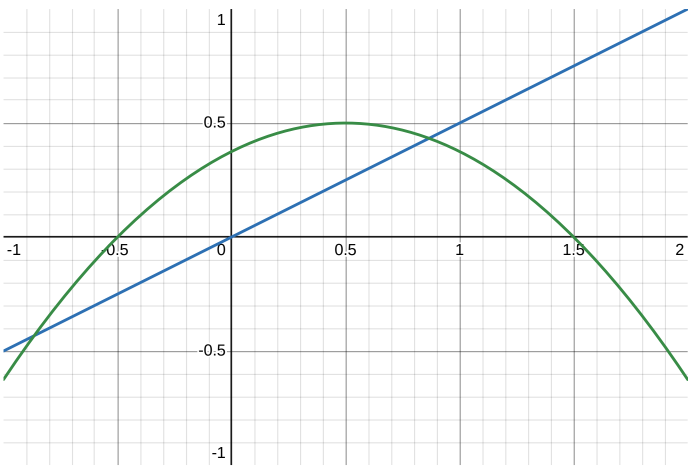
By

A continuous function from a compact metric space to $\mathbb{R}$ has a global maxima *and* a global minima.

we already know any continuous function on $[a,b]$ has a global maxima and a global minima.

There are two possibilities

- at least **one extremum is in the interior** (a,b)
- maxima and the minima **both extremums are at the boundary points** a, b



Obviously it is possible to have $f' \neq 0$ on $[a,b]$ for a continuous function on $[a,b]$ that is differentiable function on (a,b), just take $f(x) = x$ on $[0,1]$.

$\mathcal{C}[a,b]$ functions with fixed endpoints differentiable on (a,b) have at least one stationary point

But what if we demand the endpoints have the same value. This defines a function

$$\begin{matrix} f : [a,b] \to \mathbb{R} \\ f(a) = f(b) \end{matrix} \leftrightarrow f : S^1 \to \mathbb{R}$$

on the circle

$$\frac{[a,b]}{a \sim b} \cong_{\text{Man}} S^1$$

but the hypothesis of f being differentiable on (a,b), this just says f is differentiable on S^1 except at one point.

(Rolle's theorem extended by Cauchy) Let a continuous

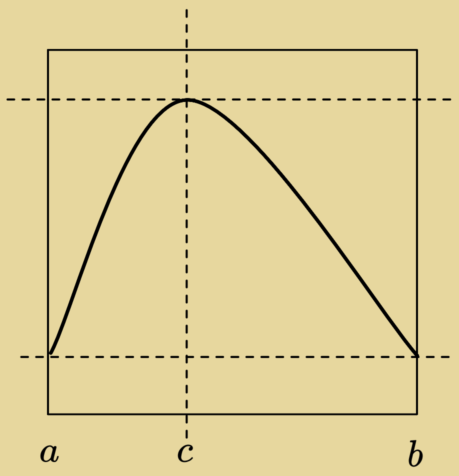
$$\begin{matrix} f : [a,b] \to \mathbb{R} \\ f(a) = f(b) \end{matrix}$$

such that

$$f' : (a,b) \to \mathbb{R} \cup \{\infty, -\infty\}$$

exists. Then **there is a point in the interior where the derivative is zero**

$$\exists c \in (a,b) \text{ such that } f'(c) = 0$$



- **(Extremum value theorem for continuous $[a,b] \to \mathbb{R}$)** Let

  $$f : [a,b] \to \mathbb{R}$$

  be **continuous**, then a global maxima *and* a minima exists $M, m \in [a,b]$ for f.

  By

  A continuous function from a compact metric space to $\mathbb{R}$ has a global maxima *and* a global minima.

- Define M, m such that $f(M) = \sup f$ and $f(m) = \inf f$
  - If one of M, m is in the interior (a, b) then $f'(m) = 0$ or $f'(M) = 0$ at that extremum value because of

    > **(Derivative, if it exists, must be 0 at local extremas)** Let a function $f : (a, b) \to \mathbb{R}$ have a local maxima or minima at $c \in (a, b)$. If f has an extended derivative at c then the derivative must be zero
    >
    > $$f'(c) \in \mathbb{R} \cup \{-\infty, \infty\} \implies f'(c) = 0$$
    >
    > Therefore,
    >
    > $$\{\text{local extremum of } f\} \subseteq Z(f')$$

  - If $M, m \in \{a, b\}$, then

    $$f(a) = f(b) \implies M = m$$

    so the minima and maxima values are equal so the function must be a constant: $f(a) = f(b) = f(x)$ for all $x \in (a, b)$. A constant function is always differentiable and derivative is zero: $f'(x) = 0$ for all $x \in (a, b)$.

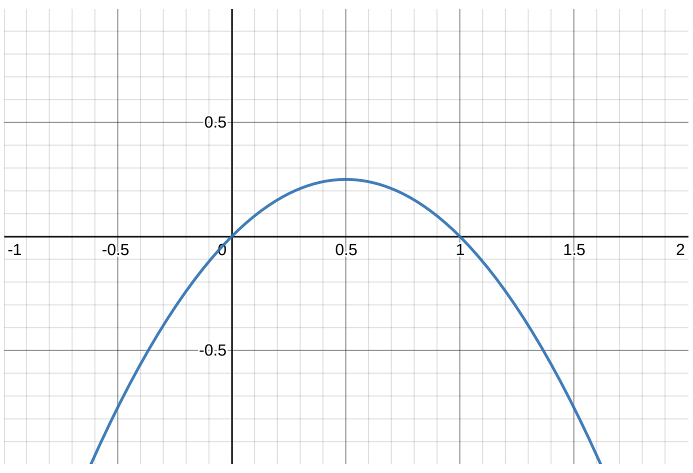
- From the proof of Rolle's theorem, we know more about the point c, as we may choose c to be a global maximum or a global minimum for a non-constant f.
- **If f is non-constant**, global max and min must be necessarily have different values and are on two distinct points M, m where

$$f(M) := \sup f[a, b] \neq f(m) := \inf f[a, b]$$

- In general **both** points would not be in the interior or have derivative zero by the example of $x(1 - x)$ on $[0, 1]$.

Example

$x(1 - x)$ on $[0, 1]$



mean value theorems

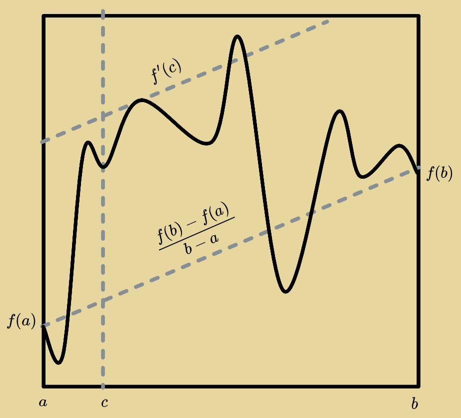
Even though the statement does not contain all the information from the proof, still the statement of Rolle's has consequences.

(Lagrange's mean value theorem) Let a continuous function

$$f : [a, b] \to \mathbb{R}$$

be differentiable on (a, b). Then there is a point $c \in (a, b)$ at which

$$f(b) - f(a) = (b - a) f'(c)$$

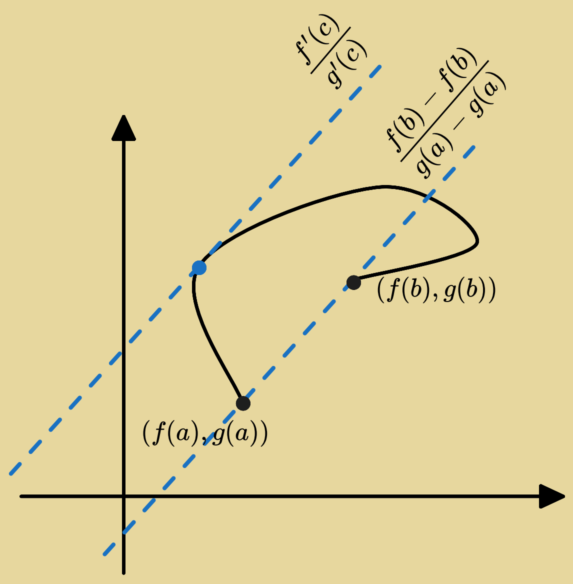


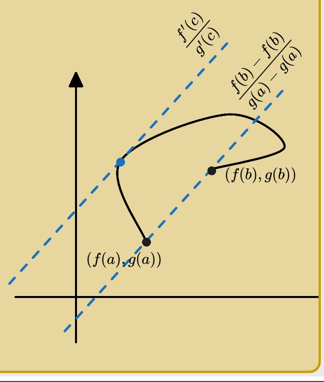
(Cauchy's mean value theorem) Let two continuous

$$(f, g) : [a, b] \to \mathbb{R}^2$$

be differentiable on (a, b). Then there is a point $c \in (a, b)$ at which

$$(f(b) - f(a)) g'(c) = (g(b) - g(a)) f'(c)$$



		applying Rolles' on
Cauchy's mean value theorem	**(Cauchy's mean value theorem)** Let two continuous $(f,g):[a,b]\to\mathbb{R}^2$ be differentiable on $(a,b$ Then there is a point $c\in(a,b)$ at which $(f(b)-f(a))g'(c)=(g($ 	the difference $h(t):=(f(b)-f(a))g(t)-(g($ on $t\in[a,b]$, Then h is continuous on $[a,b]$ and differentiable in (a,b) and $h(a)=f(b)g(a)-f(a)g(b)=$
Lagrange's mean value theorem		In the *generalized mean value theorem* we put $g(t)=t$ $h(t):=(f(a)-f(b))t-(b-a$ then $h(a)=f(b)a-f(a)b=h(b)$
	$(f(b)-f(a))c=\frac{1}{2}(b^2-a^2)f$	put $g(t)=\frac{t^2}{2}$ we have
for determinants	Let three continuous $g_1,g_2,g_3:[a,b]\to\mathbb{R}$ be differentiable on (a,b). Then for $D(x):=\det\begin{bmatrix}g_1(x) & g_2(x) & g \\ g_1(a) & g_2(a) & g \\ g_1(b) & g_2(b) & g\end{bmatrix}$	We observe $D'(x)=\det\begin{bmatrix}g_1'(x) & g_2'(x) & g \\ g_1(a) & g_2(a) & g \\ g_1(b) & g_2(b) & g\end{bmatrix}$ and $D(a)=0=D(b)$ thus we apply *Rolle's mean value theorem* on D itself.

monotonicity

> **Corollary of mean value theorem:** If
>
> $$f:(a,b)\to\mathbb{R}$$
>
> is differentiable, then for any proper subinterval $(x_1,x_2)\subset(a,b)$ by mean value theorem we have
>
> $$\exists x\in(x_1,x_2):f(x_2)-f(x_1)=(x_2-x_1)f'(x)$$

Thus

Hypothesis on f'	Consequence of **Corollary of mean value theorem: If** $f:(a,b)\to\mathbb{R}$ **is differentiable, then for any proper subinterval** $(x_1,x_2)\subset(a,b)$ **by** mean value theorem **we have** $\exists x\in(x_1,x_2):f(x_2)-f(x_1)=(x_2-x_1).$
$f'\geq 0$ on (a,b)	$\forall x_1<x_2\in(a,b),f(x_1)-f(x_2)\geq 0$ thus f is **increasing** on (a,b)
$f'=0$ on (a,b)	$\forall x_1<x_2\in(a,b),f(x_1)-f(x_2)=0$ thus f is **constant** on (a,b)
$f'\leq 0$ on (a,b)	$\forall x_1<x_2\in(a,b),f(x_1)-f(x_2)\leq 0$ thus f is **decreasing** on (a,b)
$f'<0$ on (a,b)	$\forall x_1<x_2\in(a,b),f(x_1)-f(x_2)<0$ thus f is **strictly decreasing** on (a,b)
$f'>0$ on (a,b)	$\forall x_1<x_2\in(a,b),f(x_1)-f(x_2)>0$ thus f is **strictly increasing** on (a,b)
$f'=\alpha$ is a constant on (a,b) or $f''=0$ on (a,b)	$f(x)-\alpha x$ has zero derivative, thus f is **linear** $f(x)\equiv f'(0)x+f(0)$

Hypothesis on $$f''$$ **or higher derivatives**	**Consequence of**
	Corollary of mean value theorem: If $$f:(a,b)\to\mathbb{R}$$ **is differentiable, then for any proper subinterval** $(x_1,x_2)\subset(a,b)$ **by** mean value theorem **we have** $$\exists x\in(x_1,x_2): f(x_2)-f(x_1)=(x_2-x_1).$$
$f'=\alpha$ is a constant on (a,b) or $$f''=0$$ on (a,b)	$f(x)-\alpha x$ has zero derivative, thus f is **linear** $$f(x)\equiv f'(0)x+f(0)$$
$$f''\geq 0$$ on (a,b)	$$f\left(\frac{x+y}{2}\right)\leq\frac{f(x)+g(x)}{2}$$ thus f is **convex** on (a,b)
$$f''\leq 0$$ on (a,b)	$$f\left(\frac{x+y}{2}\right)\geq\frac{f(x)+g(x)}{2}$$ thus f is **concave** on (a,b)
$$f^{(n)}=0$$ on (a,b)	f is a **polynomial** $$f(x)=\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}x^k$$

intermediate value property of derivatives

Let

$$f:[a,b]\to\mathbb{R}$$

be differentiable and $\lambda\in(f'(a),f'(b))$. Then there is a $c\in[a,b]$ such that

$$f'(c)=\lambda$$

bounded derivative and uniform continuity

Let

$$f:U(\text{interval})\subseteq\mathbb{R}\to\mathbb{R}$$

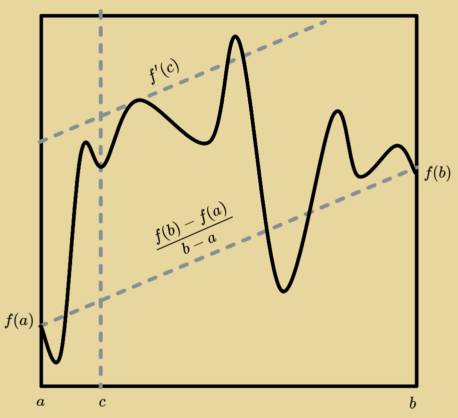
have a bounded derivative $|f'|\leq M$ for some $M\geq 0$. Then for every $x,y\in U$ by

(Lagrange's mean value theorem) Let a continuous function

$$f:[a,b]\to\mathbb{R}$$

be differentiable on (a,b). Then there is a point $c\in(a,b)$ at which

$$f(b)-f(a)=(b-a)f'(c)$$



there is a $c\in(x,y)$ such that

$$\begin{aligned} f(y)-f(x)&=(y-x)f'(c)\\ |f(y)-f(x)|&=|y-x|\,|f'(c)|\\ &\leq M\,|y-x| \end{aligned}$$

So for any $\epsilon>0$ choosing $\delta:=\epsilon/M$ we have

$$x,y\in U,|y-x|\leq\delta=\frac{\epsilon}{M}\implies|f(y)-f(x)|\leq M\frac{\epsilon}{M}=\epsilon$$

Hence, f is **uniformly continuous** on U.

$\mathcal{C}^1\implies$ locally Lipschitz

Let

$$f:[a,b]\to\mathbb{R}$$

be differentiable and

$$f':[a,b]\to\mathbb{R}$$

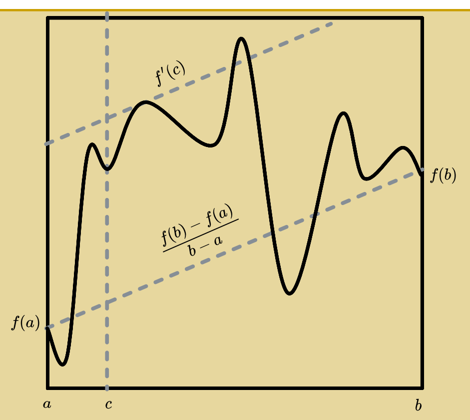
be continuous. Then f' is bounded, say $|f'|\leq M$. Then for any $x,y\in[a,b]$ by

(Lagrange's mean value theorem) Let a continuous function

$$f:[a,b]\to\mathbb{R}$$

be differentiable on (a,b). Then there is a point $c\in(a,b)$ at which

$$f(b)-f(a)=(b-a)f'(c)$$



we have

$$\exists c \in (a,b) : \left|\frac{f(y)-f(x)}{y-x}\right| = |f'(c)| \leq M$$

Hence, f is **Lipshitz continuous** on $[a,b]$.

bounded derivative $\iff$ Lipshitz

Let

$$f : U(\text{interval}) \subseteq \mathbb{R} \to \mathbb{R}$$

be differentiable.

- Then f being M-Lipshitz implies

$$\left|\frac{f(y)-f(y+h)}{h}\right| \leq M \implies |f'(y)| \leq M$$

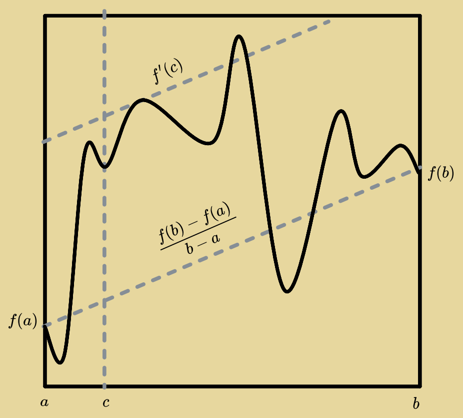
  so f has a bounded derivative on U.
- Let f have a bounded derivative on U, $|f'| \leq M$. Then for any $x, y \in U$, by

  (Lagrange's mean value theorem) Let a continuous function

$$f : [a,b] \to \mathbb{R}$$

  be differentiable on (a,b). Then there is a point $c \in (a,b)$ at which

$$f(b) - f(a) = (b-a)f'(c)$$



  we have

$$\exists c \in (x,y) : \left|\frac{f(y)-f(x)}{y-x}\right| = |f'(c)| \leq M$$

  Thus, f is M-**Lipschitz**.

differentiable functions on the circle

Bug

If we have a differentiable function on the circle S^1, that is equivalent to a continuous

$$\begin{aligned} f : [a,b] &\to \mathbb{R} \\ f(a) &= f(b) \end{aligned}$$

which is differentiable on (a,b) and the left hand derivative at $b =$ right hand derivative at a.
Then such a non-constant f has distinct maxima and minima

If we have a differentiable non-constant function on S^1 then we have

$$\begin{aligned} f(M) := \sup f[a,b] \neq f(m) &:= \inf f[a,b] \\ f'(M) = 0 &= f'(m) \end{aligned}$$

mean value for $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$

- Let

$$f : U \text{ (open)} \subseteq \mathbb{R}^n \to \mathbb{R}$$

  be differentiable

$$\mathrm{d}f : U \to \mathbb{R}^{n*}$$

  - For any $x, y \in U$ such that the line segment $\overline{xy} \subseteq U$ we have

$$\begin{aligned} f \circ \overline{xy} : [0,1] &\to \mathbb{R} \\ t &\mapsto f((1-t)x + ty) \end{aligned}$$

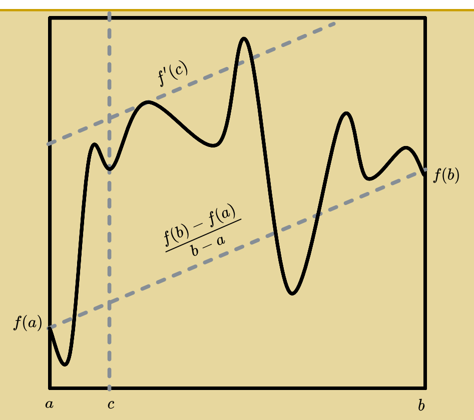
  - By

    (Lagrange's mean value theorem) Let a continuous function

$$f : [a,b] \to \mathbb{R}$$

    be differentiable on (a,b). Then there is a point $c \in (a,b)$ at which

$$f(b) - f(a) = (b-a)f'(c)$$



there is a $c \in (0,1)$ such that

$$\begin{aligned} f(y) - f(x) &= \frac{\mathrm{d}}{\mathrm{d}t} f((1-t)x + ty)\Big|_{t=c} \\ &= \mathrm{d}_{(1-c)x+cy} f(y-x) \end{aligned}$$

- This implies

$$|f(y) - f(x)| \le \left\|\mathrm{d}_{(1-c)x+cy} f\right\|_{\mathbb{R}^{n*}} |y - x|$$

for the operator norm on $\mathbb{R}^{n*}$.

Let

$$f : U \text{ (open, convex)} \subseteq \mathbb{R}^n \to \mathbb{R}^m$$

be differentiable

$$\mathfrak{D} f : U \to \text{EndVec}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m)$$

such that the operator norm of the derivative on U is bounded by M

$$\forall x \in U, \|\mathfrak{D}_x f\|_{n \to m} \le M$$

Then f is M-Lipshitz on U

$$\forall x, y \in U, |f(y) - f(x)| \le M\,|y - x|$$

Example

bounded derivative but non-Lipshitz

The operator norm of the derivative

$$\|\mathrm{d}\theta\| = \frac{\sqrt{x^2 + y^2}}{x^2 + y^2} = \frac{1}{\sqrt{x^2 + y^2}}$$

is bounded on complement of the unit disk

$$\|\mathrm{d}\theta\| \le 1 \text{ on } \left(\mathbb{R}^2 \setminus (-\infty, 0)\right) \setminus \overline{D^2}$$

However,

$$\frac{|\theta(-2,\epsilon) - \theta(-2,-\epsilon)|}{|(-2,\epsilon) - (-2,-\epsilon)|} \xrightarrow{\epsilon \to 0} \frac{2\pi}{2\epsilon}$$

Therefore,

$$|\theta(y) - \theta(x)| \le \underbrace{\|\mathrm{d}\theta\|}_{\le 1} |y - x|$$

does not hold.

[1]

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1. https://math.stackexchange.com/questions/704108/a-generalization-of-the-mean-value-theorem ↩

Current note has 1 direct children and 1 total descendants.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $\Gamma\backslash\mathbb{R}^n, \Gamma\backslash\mathbb{C}^n$
    - mapping
      - Mean value property of functions
        - Taylor's theorem

And it has 5 siblings.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $\Gamma\backslash\mathbb{R}^n, \Gamma\backslash\mathbb{C}^n$
    - mapping
      - Image of balls under differentiable maps
      - Inverse of differentiable maps
      - Mean value property of functions
      - Operator norm of linear maps between $\mathbb{R}^n$
      - Sequence of functions on $\mathbb{R}^d$