Info

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Rotational symmetrization of holomorphic functions

squishy.pow.formal power-only get multiplies of N

Only sum terms that are multiples of N in $\mathbb{C}[[z]]$

#wiki/genfx

- Let

$$f(z) = \sum_{n \geq 0} a_n z^n$$

  be a formal power series in complex entries, that is $f(z) \in \mathbb{C}[[z]]$.
- Fix a positive integer N. Let ζ_N be a N-th root of unity.
  - Then if $n \in N\mathbb{N}$ then

$$\sum_{k=1}^{N} \zeta_N^{nk} = \sum_{k=1}^{N} (1)^k = N$$

  - If $n \in \mathbb{N} \setminus N\mathbb{N}$ then

$$\sum_{k=0}^{N-1} (\zeta_N^n)^k = 1 + \zeta_N + \zeta_N^2 + \dots + \zeta^{N-1} = \frac{1 - \cancelto{1}{\zeta^{nN}}}{1 - \zeta^n} = 0$$

- Then

$$\begin{aligned}
f(z) &= a_0 + a_1 z + a_2 z^2 + \dots \\
f(\zeta_N z) &= a_0 + a_1 \zeta_N z + a_2 \zeta_N^2 z^2 + \dots \\
f(\zeta_N^2 z) &= a_0 + a_1 \zeta_N^2 z + a_2 \zeta_N^4 z^2 + \dots \\
f(\zeta_N^3 z) &= a_0 + a_1 \zeta_N^3 z + a_2 \zeta_N^6 z^2 + \dots \\
\dots & \\
f(\zeta_N^{N-1} z) &= a_0 + a_1 \zeta_N^{N-1} z + a_2 \zeta_N^{2N-2} z^2 + \dots
\end{aligned}$$

  - which means

$$\begin{aligned}
\sum_{k=1}^{N} f(\zeta_N^k z) &= \sum_{n \geq 0} a_n \left(\sum_{k=1}^{N} \zeta_N^{nk} \right) z^n = \sum_{l \geq 0} a_{lN} N z^{lN} \\
\frac{1}{N} \sum_{k=1}^{N} f(\zeta_N^k z) &= \sum_{l \geq 0} a_{lN} z^{lN} \\
&= a_0 + a_N z^N + a_{2N} z^{2N} + \dots
\end{aligned}$$

  - Moreover

$$\begin{aligned}
\frac{1}{N} \sum_{k=1}^{N} f(\zeta_N^k \sqrt[N]{z}) &= \sum_{l \geq 0} a_{lN} z^l \\
&= a_0 + a_N z + a_{2N} z^2 + \dots
\end{aligned}$$

Then if $f : U \to \mathbb{C}$ is holomorphic on a open subset U of $\mathbb{C}$ and

$$f(z) = \sum_{n \geq 0} a_n z^n$$

on a disk, then call

$$\frac{1}{N} \sum_{k=1}^{N} f(\zeta_N^k z) = \sum_{l \geq 0} a_{lN} z^{lN}$$

the $\mathbb{Z}_N$**-symmetrizer of** f.

> **Proposition:** The $\mathbb{Z}_N$-symmetrizer of f is invariant under the $\mathbb{Z}_N$ action on $\mathbb{C}$.

Call the function

$$\frac{1}{N} \sum_{k=1}^{N} f(\zeta_N^k \sqrt[N]{z}) = \sum_{l \geq 0} a_{lN} z^l = a_0 + a_N z + a_{2N} z^2 + \dots$$

the **rooted** $\mathbb{Z}_N$**-symmetrizer of** f which is also holomorphic $U \to \mathbb{C}$.

Example

The rooted $\mathbb{Z}_2$ rotational symmetrizer of $\cos z$ is $\cos \sqrt{z}$.

Current note has 0 direct children and 0 total descendants.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma \backslash^{\mathbb{R}^n}, _\Gamma \backslash^{\mathbb{C}^n}$
    - Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - Rotational symmetrization of holomorphic functions

And it has 19 siblings.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma \backslash^{\mathbb{R}^n}, _\Gamma \backslash^{\mathbb{C}^n}$
    - Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - Approximation of holomorphic functions on $\mathbb{C}$ by rational functions

- Embedding unit disk D into $\mathbb{C}^3$
- Factorization of holomorphic functions on $\mathbb{C}$
- Global holomorphic functions
- Equivalent descriptions of holomorphic functions
- Meromorphic functions on the complex plane
- Holomorphic functions meromorphic at infinity
- Modular forms
- Reconstructing meromorphic functions from stalks
- Extending holomorphic functions by reflections
- Rotational symmetrization of holomorphic functions
- Sheaf of holomorphic functions on $\mathbb{C}$
- $\mathscr{O}(U)$
- $\mathscr{O}(\mathbb{C})$
- $\mathscr{O}(D)$
- $\mathscr{O}^p(H^2_{\mathrm{U}})$
- $\mathscr{O} \cap L^p$
- $\mathscr{O}(S^1)$
- Zeros and singularities of holomorphic functions