Info

This note, that was found here, as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
was created on January 24, 2025 3:11:22 PM,
was last modified on May 17, 2026 7:36:36 PM, and
was exported on September 7, 2026 3:04:08 PM.

Construction of $\mathbb{Q}_{\hat{p}}$

absolute values on $\mathbb{Q}$

Definition. Absolute value on $\mathbb{Q}$

An **absolute value on** $\mathbb{Q}$ is a map

$$| \ |_* : \mathbb{Q} \to \mathbb{R}_{\geq 0}$$

that satisfies

- $|0|_* = 0$ and $x \neq 0 \implies |x|_* > 0$
- $| \ |_* : U(\mathbb{Q}) = \mathbb{Q} \setminus \{0\} \to \mathbb{R}_{>0}$ is a group homomorphism
- the *triangle inequality*

$$|x+y|_* \leq |x|_* + |y|_*$$

- By second condition we have

$$|-x|_* = |x|$$

  - So

$$|-1|_*^2 = |1|_* = |1|^2 \in \{-1, 1\}$$

  - And thus $|1|_* = 1 = |-1|_*$
- By second condition we have

$$|pq|_* = |p|_* |q|_*$$

  and

$$\left|\frac{p}{q}\right|_* = \frac{|p|_*}{|q|_*}$$

  - Hence, the absolute value is determined by what values it takes on naturals ≥ 2

$$\{2, 3, \ldots\} = \mathbb{N} + 2$$

  - Even better, the absolute value is determined by its values on primes

$$\{2, 3, 5, 7, 11, 13, \ldots\}$$

- **Proposition:**

$$a \in \mathbb{N} \implies |a|_* \leq a$$

non-Archimedian case

- Fix an absolute value $| \ |_*$ such that $\exists a \in \mathbb{N}, a > 1 : |a|_* \leq 1$.
  - We write $b \in \mathbb{N}$ in "base a" (as $a > 1$) as

$$b = \sum_{k=0}^{m} c_k a^k$$

    for $0 \leq c_k < a$ and $c_m \neq 0$.
  - By *triangle inequality*

$$|b|_* \leq \sum_{k=0}^{m} \underbrace{|c_k|_*}_{\leq c_k < a} \underbrace{\left|a^k\right|_*}_{=|a|_*^k \leq 1} < (m+1)a$$

  - As $c_m \neq 0$ means

$$\begin{aligned}
&(1 - c_m)a^m \leq b - c_m a^m < (a - c_m)a^m \\
&a^m - c_m a^m \leq b - c_m a^m < a^{m+1} - c_m a^m \\
\implies &a^m \leq b < a^{m+1} \\
\implies &m \log(a) \leq \log(b) \leq (m+1)\log(a)
\end{aligned}$$

    Thus

$$|b|_* < \left(\frac{\log(b)}{\log(a)} + 1\right)a$$

  - Replacing b with b^n we have

$$|b|_*^n = |b^n|_* < \left(n\frac{\log(b)}{\log(a)} + 1\right)a$$

    for all $n \geq 1$.
  - But if $|b|_*^n > 1$ then $|b|_*^n$ cannot grow *at most* linearly as

$$x > 1 \implies x^n > n\alpha_0 + \alpha_1 \text{ eventually}$$

  - Hence, if there is one natural a with absolute value ≤ 1 then the absolute value on all naturals is ≤ 1

$$\exists a \in \{2, 3, \ldots\}, |a|_* \leq 1 \implies \forall b \in \{2, 3, \ldots\}(|b|_* \leq 1)$$

- Assume the absolute value is not *trivial*. Then there is $n \in \{2, 3, \ldots\}$ such that $|n|_* \neq 1$. This would mean $|n|_* < 1$. Write

$$n = p_1^{a_1} \dots p_k^{a_k}$$
$$\implies 1 > |n|_* = |p_1|_*^{a_1} \dots |p_k|^{a_k}$$

- If all primes had $|p|_* \geq 1$ then

  $$1 > |n|_* = |p_1|_*^{a_1} \dots |p_k|^{a_k} \geq 1$$

  Thus there is at least one prime p with $|p|_* < 1$.
- *Assume* there are two distinct primes p, q with $|p|_* < 1, |q|_* < 1$.
  - Then

    $$|p|_*^m < \frac{1}{2}, |q|_*^l < \frac{1}{2}$$

Definition. p-adic norm $|\,|_p : \mathbb{Q} \to \mathbb{R}_{\geq 0}$

For a prime $p \in \mathbb{N}$, the p-adic norm on $\mathbb{Q}$ is defined by defining

$$|p|_p = \frac{1}{p}$$

and

$$|q|_p = 1$$

for all other primes $q \neq p$.

Current note has 0 direct children and 0 total descendants.

- stretchy
  - $\mathbb{Q}_{\hat{p}}$
    - Construction of $\mathbb{Q}_{\hat{p}}$

And it has 1 siblings.

- stretchy
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