Info

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Hamiltonian Lie group action on a symplectic manifold

For $\mathbb{R}$,

$$\{\text{symplectic } \mathbb{R} \curvearrowright (M,\omega)\} \underset{\exp(t\,-)}{\overset{\frac{\mathrm{d}}{\mathrm{d}t}}{\rightleftharpoons}} \{\text{complete } \mathrm{SymVec}(M,\omega)\}$$

The action is *"Hamiltonian"* if there exists a $H \in \mathcal{C}^\infty(M)$ that is, a smooth map

$$H : M \to \mathbb{R}$$

such that its Hamiltonian vector field generates the $\mathbb{R}$ action

$$\omega^{-1}\mathrm{d}H = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1} \Phi^t$$

Given a symplectic action

Definition. Symplectic Lie group action on a symplectic manifold

A smooth Lie group action

$$\Phi : G \curvearrowright M$$

on a symplectic manifold is a **symplectic action** if G acts by symplectomorphisms

$$\Phi : G \to \mathrm{Sym}(M,\omega)$$

we have a Lie algebra homomorphism

$$\mathrm{d}\Phi : \mathfrak{g} \to \mathrm{SymVec}(M,\omega)$$

If we demand the image is inside Hamiltonian vector fields

$$\mathrm{d}\Phi : \mathfrak{g} \to \mathrm{HamVec}(M,\omega)$$

then we have

Definition. Hamiltonian Lie group acition on a symplectic manifold

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Definition. Symplectic Lie group action on a symplectic manifold

A smooth Lie group action

$$\Phi : G \curvearrowright M$$

on a symplectic manifold is a **symplectic action** if G acts by symplectomorphisms

$$\Phi : G \to \mathrm{Sym}(M,\omega)$$

is a **Hamiltonian action** if there exists a smooth map

$$\begin{aligned} \mu : M &\to \mathfrak{g}^* \\ p &\mapsto \mu^{(-)}(p) \end{aligned}$$

such that

- for each $v \in \mathfrak{g}$, the component of μ along v

  $$\mu^v(-) \in \mathcal{C}^\infty(M)$$

  is the function whose Hamiltonian vector field that generates the action by v

  $$\omega^{-1}\mathrm{d}\mu^v = \mathrm{d}_i\Phi(v) \iff \mathrm{d}\mu^v = \omega(\mathrm{d}_i\Phi(v), -)$$

- μ is equivariant with the G action on M and coadjoint action on $\mathfrak{g}^*$

  $$\mu \circ \Phi^g = \mathrm{Ad}^*_g \circ \mu$$

For a basis (v_i) of $\mathfrak{g}$, we have $\dim G$ functions

$$\mu^{v_i} : M \to \mathbb{R}$$

whose Hamiltonian vector field = vector field that generates v_i-actions

$$\omega^{-1}\mathrm{d}\mu^{v_i} = \mathrm{d}\Phi(v)$$

such that the map $\mu : M \to \mathfrak{g}^*$ is equivariant.

So given the moment map we have the G-action

$$\exp t(\omega^{-1}\mathrm{d}\mu^{v_i})$$

Definition. A **symplectic Hamiltonian G-manifold** is a Hamiltonian G-action

$$\Phi : G \underbrace{\curvearrowright}_{\mu} (M,\omega)$$

with moment map μ.

symmetry

On a

> **Definition.** A **symplectic Hamiltonian G-manifold** is a Hamiltonian G-action
>
> $$\Phi : G \underbrace{\curvearrowright}_{\mu} (M, \omega)$$
>
> with moment map μ.

a function

$$f \in \mathcal{C}^{\infty}(M)$$

is G-**invariant** $\iff$ μ is a **constant on the flow** of the Hamiltonian vector field of f.

$$L_{X_f}\mu^v = \mathrm{d}\mu^v(X_f) = -\omega(\mathrm{d}\Phi(v), X_f) = -L_{\mathrm{d}\Phi(v)}f$$

[1]

- We have a correspondence between G-**invariant functions** f

  $$\iff$$

  $\mathbb{R}$-*flows* $\exp tX_f$ on M such that μ is constant on the flow

  $$\iff$$

  $\mathbb{R}$-*flows* $\exp tX_f$ on M that commute with the action of $\Phi : G \curvearrowright M$

  $$0 = \omega(\mathrm{d}\Phi(v), X_f) = -[\mathrm{d}\Phi(v), X_f]$$

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1. Ana Cannas da Silva - Lectures on Symplectic Geometry -Springer (2009).pdf > page=158 ↩
2. Lec08.pdf (ustc.edu.cn) ↩