Info

This note, that was found here, as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
was created on September 18, 2023 8:02:16 PM,
was last modified on September 7, 2026 1:11:16 PM, and
was exported on September 7, 2026 3:09:58 PM.

Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1

Definition. Holomorphic functions on $\mathbb{C}$

Given an open subset $U \subseteq \mathbb{C}$, a function

$$f : U \to \mathbb{C}$$

is called **holomorphic** if it satisfies any of the equivalent conditions

- $$\lim_{h \to 0, h \in \mathbb{C} \setminus \{0\}} \frac{f(z+h) - f(z)}{h}$$
  exists for all point $z \in U$
- $\iff$ f is *analytic* that is $\forall z_0 \in U, \exists \epsilon > 0$ such that
  $$f(z) = \sum_{n \geq 0} a_n (z - z_0)^n$$
  for all $z \in B(\epsilon, z)$
- $\iff$ f satisfies the Cauchy-Riemann equations
  $$\frac{\partial}{\partial \bar{z}} f = 0$$
- $\iff$ it satisfies Cauchy integral formula, that is, $f \in C^1(U)$, for all $z_0 \in U$, $\exists \epsilon > 0$
  $$f(z_0) = \frac{1}{2\pi i} \int_{\partial B(\epsilon, z_0)} \frac{f(z)}{z - z_0}$$
- $\iff$ the complexified total derivative (whose Jacobian is)
  $\mathfrak{D}^{\mathbb{C}}_{z_0} f : T_{z_0}\mathbb{R}^2 \otimes \mathbb{C} \to T_{f(z_0)}\mathbb{R}^2 \otimes \mathbb{C}$
  $$\begin{bmatrix} \frac{\partial}{\partial z} f & \frac{\partial}{\partial \bar{z}} f \\ \underbrace{\frac{\partial}{\partial z} \bar{f}}_{\overline{\frac{\partial}{\partial \bar{z}} f}} & \underbrace{\frac{\partial}{\partial \bar{z}} \bar{f}}_{\overline{\frac{\partial}{\partial z} f}} \end{bmatrix}$$
  is diagonal $\iff$ it is a $\mathbb{C}$-linear map and then we have the complex total derivative (whose Jacobian is)
  $$\left[\frac{\partial}{\partial z} f\right]$$

- $\mathscr{O}(U)$
- Sheaf of holomorphic functions on $\mathbb{C}$
- Derivative of holomorphic functions
- Integral of holomorphic differential forms
- Equivalent descriptions of holomorphic functions

on a	antiderivative	local	global
any open set	no	non-constant holomorphic function locally looks like w^k on a change of coordinates	
connected open sets	no		
space.R.2.disk.Hol	yes	**(Open mapping theorem for open subsets in $\mathbb{C}$)** Let f be *non-constant* holomorphic function on a connected open set U then $f(U)$ is open (and connected).	**(Maximum of holomorphic functions on is never attained on the interior)** Let $f \in \mathscr{O}(U)$ be a **non-constant** holomorphic function on a open set $U \subseteq \mathbb{C}$. Then $\lvert f \rvert$ has no local maximum on U. For $f \in \mathscr{O} \cap \mathcal{C}(\overline{U})$ where $U \subseteq \mathbb{C}$ is a bounded open subset, $\lVert f \rVert_\infty$ is attained on ∂U.
open simply connected (biholomorphic to a disk)	yes		

<table>
<tr><th>on a</th><th>antiderivative</th><th>local</th><th>global</th></tr>
<tr><td>upper half plane (biholomorphic to a disk)</td><td>yes</td><td></td><td></td></tr>
<tr><td>punctured disks
Holomorphic functions on punchered disk</td><td>no</td><td>Proposition: Let $z_0 + RD^* := \{z : 0$
and
$f : z_0 + RD^* \to \mathbb{C}$
be holomorphic, then it has a Laurent series
$$f(z) = \sum_{n \in \mathbb{Z}} a_n(z -$$
which converges in $a + RD^*$ (that means each of the series $\sum_{n \geq 0} a_n z^n, \sum_{n \geq 1}$ converges near $z = 0$) given by
$$a_n = \frac{1}{2\pi i} \int_{S^1_s} \frac{}{(z -}$$</td><td>
<table>
<tr><th>f is holomorphic on $z_0 +$</th><th>type of (isolated) singularities</th><th>the Laurent series $f(z)$</th><th>image of f on $z_0 +$</th></tr>
<tr><td></td><td>removable singularity, holomorphic at z_0</td><td>$c_n =$ for all $n <$</td><td>bounded</td></tr>
<tr><td>$(z -$ has a removable singularity for some $k >$ but f does not</td><td>pole, meromorphic at z_0</td><td>eventually $c_n =$</td><td>$\|f(z$ as $\|z\|$</td></tr>
<tr><td>$(z -$ does not have a removable singularity for any $k \in$</td><td>essential singularity</td><td>c_n does not become 0 eventually</td><td>image is dense on $\mathbb{C}$ (actually reaches almost all values on $\mathbb{C}$)</td></tr>
</table>
</td></tr>
<tr><td>annulus of width R</td><td>no</td><td></td><td></td></tr>
<tr><td>$\mathbb{C}$ $\mathscr{O}(\mathbb{C})$</td><td>yes</td><td></td><td>bounded $\implies$ constant</td></tr>
</table>

properties of holomorphic functions

- Convergent power series
- Cauchy integral formula for holomorphic functions
- Integral of holomorphic differential forms
- Non-constant holomorphic functions locally look like w^k and are open mappings
- Zeros and singularities of holomorphic functions
- Sequences of holomorphic functions
  - Approximation of holomorphic functions on $\mathbb{C}$ by rational functions
- Sheaf of holomorphic functions on $\mathbb{C}$
- factorizing, constructing functions
  - Reconstructing meromorphic functions from stalks
  - Factorization of holomorphic functions on $\mathbb{C}$
- extension
  - Extending holomorphic functions by reflections
- Meromorphic functions on the complex plane
- Holomorphic functions meromorphic at infinity

- Holomorphic function whose singularities are not isolated
- Branch points and ramification index of holomorphic functions
- Global holomorphic functions
- Biholomorphic mapping onto disk

- Rotational symmetrization of holomorphic functions

- Modular forms

- Global holomorphic functions
- polynomials
- rational functions

File (11)	title	Riemann surface
squishy.CP1 Aut	$PSL(2)(\mathbb{C}) \curvearrowright \mathbb{C}P^1 \cong_{\text{Man}} S^2$	-
squishy.pow.C1 Hol cos of sqrt	$\cos\sqrt{z}$	-
squishy.pow.C1 Hol log	The complex logarithm	squishy.covering univ C minus 1
squishy.pow.C1 Hol to n factorial	$\sum_{n\geq 0} z^{n!}$	-
squishy.pow.C1 Hol z3 - 3z	$z^3 - 3z$	-
squishy.pow.exp	The complex exponential $\exp(z)$	Holomorphic on $\mathbb{C}$
squishy.pow.exp of z+1 by z-1	$\exp\left(\frac{z+1}{z-1}\right)$	-
squishy.pow.sqrt	$\sqrt{z}$	-
squishy.pow.Weierstrass elliptic	Weierstrass elliptic function $\wp$	-
squishy.rat.C1 Hol z over z2 -1	$\frac{z}{z^2-1} = \frac{z}{(z+1)(z-1)}$	$\mathbb{C} \setminus \{-1, 1\}$
stem.Rf.1Hol.mapping.z2 at all rationals	A holomorphic map branched over $\mathbb{Q}$	-

- combinatorics - Why does this random walk generating function admit an analytic continuation? - Mathematics Stack Exchange

---

Current note has 19 direct children and 43 total descendants.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, _\Gamma\backslash^{\mathbb{C}^n}$
    - Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - Approximation of holomorphic functions on $\mathbb{C}$ by rational functions
      - Embedding unit disk D into $\mathbb{C}^3$
      - Factorization of holomorphic functions on $\mathbb{C}$
      - Global holomorphic functions
      - Equivalent descriptions of holomorphic functions
        - Cauchy integral formula for holomorphic functions
        - Derivative of holomorphic functions
        - Integral of holomorphic differential forms
        - Convergent power series
      - mapping
        - Bounding derivative of holomorphic mappings
        - Bounding holomorphic functions linearly
        - Branch points and ramification index of holomorphic functions
        - Disk in holomorphic image of disk
        - Size of holomorphic image of disks
        - Maximum and minimum of holomorphic functions
        - Fibers of holomorphic mappings
        - Non-constant holomorphic functions locally look like w^k and are open mappings
        - Biholomorphic mapping onto disk
        - Conformal radius of pointed simply connected open subsets of $\mathbb{C}$
        - Holomorphic functions on punchered disk
        - Holomorphic function whose singularities are not isolated
        - A holomorphic map branched over $\mathbb{Q}$
      - Meromorphic functions on the complex plane
      - Holomorphic functions meromorphic at infinity
      - Modular forms
      - Reconstructing meromorphic functions from stalks
      - Extending holomorphic functions by reflections
      - Rotational symmetrization of holomorphic functions
      - Sheaf of holomorphic functions on $\mathbb{C}$
      - $\mathscr{O}(U)$
        - Pre-compact subsets of $\mathscr{O}(U)$
        - Sequences of holomorphic functions
      - $\mathscr{O}(\mathbb{C})$
      - $\mathscr{O}(D)$
        - Boundary values of $\mathscr{O}(D)$

- $\mathscr{O}(\overline{D})$
- $\mathscr{O}(D) \cap \mathcal{C}(\overline{D})$
- $\mathscr{O} \cap L^2(D)$
- Reconstructing $\mathscr{O}(D)$ from its boundary values
- $\mathscr{O}^p(H^2_{\mathrm{U}})$
- $\mathscr{O} \cap L^p$
- $\mathscr{O}(S^1)$
- Zeros and singularities of holomorphic functions

And it has 39 siblings.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $\Gamma \backslash \mathbb{R}^n, \Gamma \backslash \mathbb{C}^n$
    - Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
    - Open subsets of $\mathbb{R}^n$ with $\mathcal{C}^2$ boundary
    - Complex ODEs in $\mathbb{C}$
    - Circle packing on $\mathbb{R}^2$
    - Circle packing converges to the Riemann biholomorphism
    - Bounded connected open subsets of $\mathbb{C}^n$
    - Connected circular open subsets of $\mathbb{C}^n$
    - Continuous functions on $\mathbb{R}^d$
    - Dyadic cubes
    - Curves
    - Differentiable functions
    - Differential forms on $\mathbb{R}^n$
    - Fourier-Wigner transform
    - Harmonic functions composed with conformal maps
    - Hilbert transform
    - Fourier-Cauchy-Poisson correspondence of holomorphic and harmonic functions on the unit disk and their boundary values
    - Holomorphic subsets of $\mathbb{C}^n$
    - Regular hypersurfaces in $\mathbb{R}^{2n}$
    - Orientable hypersurfaces in $\mathbb{R}^n$
    - Laplace operator on $\mathbb{R}^n$
    - $l^\infty(\mathbb{R})$
    - Lebesgue measurable subsets of and functions on $\mathbb{R}^n, T^n, S^n$
    - Lebesgue measure of boundary of open sets in $\mathbb{R}^n$
    - Local $\mathbb{R}$-algebras
    - $l^p(\mathbb{R})$
    - Metric density of subsets of $\mathbb{R}^n$
    - Monotone functions on $\mathbb{R}$
    - Cauchy integral of periodic functions
    - Polygons with integer vertices
    - Open smooth maps $U \subseteq \mathbb{R}^2 \to \mathbb{C}$
    - Discrete subgroups of $\mathbb{R}^n$
    - Discrete cocompact subgroups of $\mathbb{R}^n$, flat tori
    - Ramified germs of smooth and holomorphic functions
    - Open subsets of $\mathbb{R}^n$
    - Open subsets of $\mathbb{R}^n$ equipped with the flat metric
    - Closed subsets of $\mathbb{R}^n$ equipped with smooth functions
    - Quasi-analytic smooth functions on $\mathbb{R}$
    - Star-shaped subsets of $\mathbb{R}^n$
    - ODEs in $\mathbb{R}^n \leftrightarrow$ Vector fields in $\mathbb{R}^n$