Info

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Smooth maps onto a Lie group

Definition. Left Darboux derivative of a map onto a Lie group

Let M be a smooth manifold, G be a Lie group with Lie algebra $\mathfrak{g}$ and Maurer-Cartan form ω_G and

$$f : M \to G$$

a smooth map.
Then the **left Darboux derivative** of f is the $\mathfrak{g}$-valued 1-form

$$\omega_f := f^*\omega_G \in \Omega^1(M, \mathfrak{g})$$

It is the composition

$$TM \xrightarrow{\mathfrak{D}f} TG \xrightarrow{\omega_f} \mathfrak{g}$$

> **Proposition:**
>
> $$\mathrm{d}\omega_f(X,Y) + [\omega_f(X), \omega_f(Y)] = 0$$

Thus, we have

$$\mathsf{EndMan}(M,G) \to \Omega^1(M,\mathfrak{g})^{\mathfrak{s}} \hookrightarrow \Omega^1(M,\mathfrak{g})$$

> **Proposition: (Uniqueness of the primitive)** Let M be a connected manifold and $f_1, f_2 : M \to G$ be smooth maps onto a Lie group G. Then their left Darboux derivative are the same
>
> $$\omega_{f_1} = \omega_{f_2}$$
>
> $\iff$ there exists an element $g \in G$ such that
>
> $$f_2 = gf_1$$

fundamental theorem

> **Proposition:** Let G be a Lie group with Lie algebra $\mathfrak{g}$ and M be a smooth manifold with a $\omega \in \Omega^1(M, \mathfrak{g})$ such that
>
> $$\mathrm{d}\omega(X,Y) + [\omega(X), \omega(Y)] = 0$$
>
> Then for each $p \in M$ there is a neighborhood U_p of p and a smooth map $f : U_p \to G$ such that
>
> $$\omega|_U = \omega_f$$

Let G be a Lie group with Lie algebra $\mathfrak{g}$, M be a smooth manifold with a $\omega \in \Omega^1(M, \mathfrak{g})$. Then ω is *left Darboux derivative* of some smooth map $M \to G$ $\iff$

$$\mathrm{d}\omega(X,Y) + [\omega(X), \omega(Y)] = 0$$

and the monodromy representation

$$\Phi_\omega : \pi_1(M, b) \to G$$

is trivial. Moreover, such an integral of ω, if it exists, is unique upto a left translation by a constant element of G.

Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - manifolds
    - $\mathbb{R}$-manifold group (Lie groups)
      - Smooth maps onto a Lie group

And it has 18 siblings.

- structures based on sets
  - manifolds
    - $\mathbb{R}$-manifold group (Lie groups)
      - Adjoint representation of a Lie group
      - Conjugation action of a Lie group on itself
      - Classification of Lie groups
      - Compact Lie groups
      - Connected Lie groups with discrete center
      - Connected Lie groups with non-surjective exponential map
      - Discrete subgroups of Lie groups
      - Smooth maps onto a Lie group
      - Lie groups with surjective exponential map
      - Maurer-Cartan form
      - Left-invariant Lagrangian on a Lie group
      - Lie algebra of a lie group
      - Bi-invariant metrics on Lie groups
      - Left invariant Riemannian metric on a Lie group
      - Semi-simple $\mathbb{R}$-Lie groups

- Almost simple groups
- Virtual subgroups of Lie groups
- Lie group with dilations