Info

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Equivalent descriptions of holomorphic functions

Definition. Holomorphic functions on $\mathbb{C}$

Given an open subset $U \subseteq \mathbb{C}$, a function

$$f : U \to \mathbb{C}$$

is called **holomorphic** if it satisfies any of the equivalent conditions

- $$\lim_{h \to 0, h \in \mathbb{C}\setminus\{0\}} \frac{f(z+h) - f(z)}{h}$$
  exists for all point $z \in U$
- $\iff$ f is *analytic* that is $\forall z_0 \in U, \exists \epsilon > 0$ such that
  $$f(z) = \sum_{n \geq 0} a_n (z - z_0)^n$$
  for all $z \in B(\epsilon, z)$
- $\iff$ f satisfies the Cauchy-Riemann equations
  $$\frac{\partial}{\partial \bar{z}} f = 0$$
- $\iff$ it satisfies Cauchy integral formula, that is, $f \in C^1(U)$, for all $z_0 \in U$, $\exists \epsilon > 0$
  $$f(z_0) = \frac{1}{2\pi i} \int_{\partial B(\epsilon, z_0)} \frac{f(z)}{z - z_0}$$
- $\iff$ the complexified total derivative (whose Jacobian is)
  $\mathfrak{D}^{\mathbb{C}}_{z_0} f : T_{z_0}\mathbb{R}^2 \otimes \mathbb{C} \to T_{f(z_0)}\mathbb{R}^2 \otimes \mathbb{C}$
  $$\begin{bmatrix} \frac{\partial}{\partial z} f & \frac{\partial}{\partial \bar{z}} f \\ \underbrace{\frac{\partial}{\partial z} \bar{f}}_{\overline{\frac{\partial}{\partial \bar{z}} f}} & \underbrace{\frac{\partial}{\partial \bar{z}} \bar{f}}_{\overline{\frac{\partial}{\partial z} f}} \end{bmatrix}$$
  is diagonal $\iff$ it is a $\mathbb{C}$-linear map and then we have the complex total derivative (whose Jacobian is)
  $$\left[\frac{\partial}{\partial z} f\right]$$

object	some derivative operators	equivalent to Cauchy-Riemann operator	Cauchy-Riemann equations is when the Cauchy-Riemann operator gives 0	integration	primitive	what does premitive satisfy
complex function $f = u + iv$	The *total derivative* $\mathfrak{D}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u_x \\ v_x \end{bmatrix}$ which splits into $\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}$	$\frac{\partial}{\partial \bar{z}} f = \frac{1}{2}\Big(($	Holomorphic		a holomorphic g such that $\frac{\partial g}{\partial z} = f$	
a (complex) differential $(1,0)$-**form** $f(z)\mathrm{d}z$		$\mathrm{d}(f(z)\mathrm{d}z) =$	closed differential form	$\int f(z)\mathrm{d}z = \int (u + iv) = \int (u\mathrm{d}x -$	an holomorphic g such that $\mathrm{d}g = f\mathrm{d}z$	
(real) differential 1-form $\omega = u\mathrm{d}x - v$ and its Hodge dual $\star\omega = v\mathrm{d}x +$		$\mathrm{d}\omega = (u_y +$ $\mathrm{d}\star\omega = (v_y -$	closed, coclosed differential form	$\int \omega + i \int$ $= \int (u\mathrm{d}x -$	a smooth ϕ such that $\mathrm{d}\phi = \omega$	$\mathrm{d} \star \mathrm{d}\phi = 0$
(real) vector field $\bar{f} = \begin{bmatrix} u \\ -v \end{bmatrix}$		$\mathrm{curl}\bar{f} = -v_x$ $\mathrm{div}\bar{f} = u_x -$	*incompressible* (area preserving/symplectic), *solenoid* vector field	$\int \bar{f} \cdot (\gamma + i\gamma)$?	a smooth ϕ such that $\bar{f} = \mathrm{grad}(\phi)$	$\Delta\phi = 0$

object	some derivative operators	equivalent to Cauchy-Riemann operator	Cauchy-Riemann equations is when the Cauchy-Riemann operator gives 0	integration	primitive	what does premitive satisfy
two real functions $u(x,y), v(x,$	Δu Δv $\mathrm{grad} u \cdot \mathrm{grad}$	$\mathrm{grad} u + \begin{bmatrix}0 \\ 1\end{bmatrix}$	gradient of u rotated 90 degrees is the gradient of v - both are harmonic and conjugate to each other			
one real function $u(x,y)$			orthogonal real (harmonic?) functions $\|\mathrm{grad} u\|_{z_0} =$ at some $z_0 \in \mathbb{C}$			

Current note has 4 direct children and 4 total descendants.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $\Gamma\backslash\mathbb{R}^n, \Gamma\backslash\mathbb{C}^n$
    - Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - Equivalent descriptions of holomorphic functions
        - Cauchy integral formula for holomorphic functions
        - Derivative of holomorphic functions
        - Integral of holomorphic differential forms
        - Convergent power series

And it has 19 siblings.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $\Gamma\backslash\mathbb{R}^n, \Gamma\backslash\mathbb{C}^n$
    - Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - Approximation of holomorphic functions on $\mathbb{C}$ by rational functions
      - Embedding unit disk D into $\mathbb{C}^3$
      - Factorization of holomorphic functions on $\mathbb{C}$
      - Global holomorphic functions
      - Equivalent descriptions of holomorphic functions
      - Meromorphic functions on the complex plane
      - Holomorphic functions meromorphic at infinity
      - Modular forms
      - Reconstructing meromorphic functions from stalks
      - Extending holomorphic functions by reflections
      - Rotational symmetrization of holomorphic functions
      - Sheaf of holomorphic functions on $\mathbb{C}$
      - $\mathscr{O}(U)$
      - $\mathscr{O}(\mathbb{C})$
      - $\mathscr{O}(D)$
      - $\mathscr{O}^p(H^2_{\mathrm{U}})$
      - $\mathscr{O} \cap L^p$
      - $\mathscr{O}(S^1)$
      - Zeros and singularities of holomorphic functions