Info

This note, that was found here, as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
was created on March 1, 2023 1:23:29 PM,
was last modified on May 17, 2026 7:36:54 PM, and
was exported on September 7, 2026 3:31:12 PM.

Tangent space of a smooth manifold

$$(T_pM, +, \ )$$

Motivation: Tangent space must contain "velocities" of smooth curves.

Motivated from the definition

> Definition. The equivalence classes of velocity
>
> Denote the quotient set of the equivalence classes of velocity by
>
> $$\text{Curves}^\infty(p \in U)/(-)'(0)$$
>
> and the equivalence classes by
>
> $$[\gamma'(0)]$$
>
> because it does not depend on chart.

we want to define a **tangent space** for a point T_pM for a point $p \in M$ which is identified with the equivalence classes of velocities.

We make definition of the tangent space as operators on smooth functions.

Definition.

$$(T_pM, +, \cdot)$$

Given a smooth manifold M,and any $p \in M$ we define the **tangent space** T_pM at the point p as

$$(T_pM, +, \cdot) \leq \text{HomVec}_{\mathbb{R}}(\mathcal{C}^\infty(M), \mathbb{R})$$

containing the operators

$$(- \circ \gamma)'(0)$$

for any smooth curve $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p$.

For any smooth manifold M, there exists an unique tangent space at the point p

> Definition.
>
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for any $p \in M$

For a n-manifold M, map

$$T_pM \to \mathbb{R}^n$$
$$(- \circ \gamma)'(0) \mapsto (\varphi \circ \gamma)'(0)$$

is an $\mathbb{R}$-linear isomorphism for any local chart φ at p. Hence,

$$\dim(T_pM) = \dim(M).$$

identifications

Tangent vectors can be identified with each of following:

- velocities of curves *in a chart*

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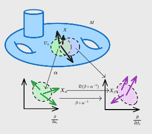
>
> $$\dim(T_pM) = \dim(M).$$

- Tangents on smooth manifolds as derivations $\mathcal{C}^\infty(M) \to \mathbb{R}$
- as an equivalence class of velocity of curves $[\gamma'(0)]$
- as *actual* velocities of curves $\gamma'(0)$
- *(for the definition itself)* directional derivative in the direction $\gamma'(0)$ at $\gamma(0)$

sett.Man.R.T.v.comparison

Comparison of constructions with tangent vectors

	$X_p \in T_pM$	T_pM	TM	$X \in \Gamma TM$	ΓTM	$\mathrm{Vec}(-)$
coordinate-free definition	$(- \circ \gamma)'(0)$	Definit Given a smooth manifold M ,and any $p \in M$ we define the **tangent space** T_pM at the point p as $(T_pM, +,$ containing the operators $(- \circ \gamma)'($ for any smooth curve $\gamma : (-\epsilon, \epsilon)$ with $\gamma(0) = p$.		Definit A **smooth/continuous/rough tangent vector field** on a smooth manifold M (with/out boundary) is a *section* of the projection $TM \to M$, that is, a *smooth/continuous/arbitrary* map $X : M \to$ satisfying $\pi_{TM} \circ X$		

	$X_p \in T_pM$	T_pM	TM	$X \in \Gamma TM$	ΓTM	$\mathrm{Vec}(-)$
in a chart		$T_pM \to \mathbb{R}^n$		*Rough* tangent vector field X on a smooth manifold M is a function $X : M \to$ $p \mapsto$ It is *continuous /smooth* if for all charts $(U_\alpha, \underbrace{\alpha}_{U_\alpha \to \mathbb{R}^{\dim}}$, $p \in U$, X_p written in the coordinate basis for T_pM given by α $X_p = \sum_{i=1}^{\dim M}$ makes the map $X_\alpha : U_\alpha \to$ $p \mapsto$ is a *continuous /smooth* map, which means $X_\alpha \circ \alpha^{-1}$ must be *continuous /smooth* map on $\alpha(U)$.		
equivalence classes of curves	$[\gamma'(0)]$					
velocities of curves	$\gamma'(t) \in T_{\gamma(t)}$	curves upto same velocity is bijective to TM	γ' is a smooth curve on TM			
curves associated with a particular chart velocity						
local basis/coordinates	$X_p = \sum_i X$	$\left\{ \frac{\partial}{\partial x_i} \vert_p \right\}$ is a basis	elements look like $(p, \sum_i X^i \frac{\partial}{\partial x}$	$X = \sum_i X^i$		
change of basis				Given the collection of *rough/continuous/smooth* maps (U_α, α) is a is a vector field on M $\Longleftrightarrow$ $(U_\alpha, \alpha), (U$ 		

	$X_p \in T_pM$	T_pM	TM	$X \in \Gamma TM$	ΓTM	$\mathrm{Vec}(-)$
action on smooth maps, equivalent to d	$X_p(f) = \mathrm{d}_p$	action of vectors on smooth maps is a derivation $X_p : \mathcal{C}^\infty(M)$		$Xf = \mathrm{d}f(X$	action of vector fields on $\mathcal{C}^\infty(M)$ is a derivation $X : \mathcal{C}^\infty(M)$	
derivations on $\mathcal{C}^\infty(M)$	$X_p : \mathcal{C}^\infty(M)$	$\mathrm{Der}_p(\mathcal{C}^\infty(M$		$X : \mathcal{C}^\infty(M)$	$\mathrm{Der}(\mathcal{C}^\infty(M)$	
$\mathcal{C}^\infty(M)$-multiplication	αX_p	$\mathbb{R}$-vector space	$\mathbb{R}$-vector bundle	$\alpha(p)X_p$	$\mathcal{C}^\infty(M)$-module	
product of derivations	X_pY_p is not a vector			XY is not a vector field		
pushforward	$F_p^*(X_p) :=$	F_p^* is a linear map	F^* is a bundle map	**when F is a diffeomorphism**	F^* is a Lie algebra map?	
functor		$\mathrm{Man}_* \to \mathrm{Ve}$ $(M,p) \mapsto T_p$	$\mathrm{Man} \to \mathrm{VecB}$ $M \mapsto TI$			man to sheafs?!
commutator	-	-	-	$[X,Y] := X$ as endomorphism in $\mathcal{C}^\infty(M)$	$\mathrm{Der}(\mathcal{C}^\infty(M)$ is a Lie $\mathbb{R}$-algebra	sheaf to Lie $\mathbb{R}$-algebras
flow of a vector field						
exponential	-	-	-	exponential of vector fields		
					By existence and uniqueness we have a global correspondence vector field	sheaf map
exponential and commutator				$\exp[tX, tY]$		
Lie derivative on $\mathcal{C}^\infty(M)$				$L_Xf = Xf$		
Lie derivative on ΓTM				$L_XY = [X,$	$L_X : \Gamma TM$ is a Lie algebra homomorphism?	
Lie derivative and pushes?						
					curves on $\Gamma TM \leftrightarrow$ time dependent vector fields	

more

- local basis of a tangent space of a smooth manifold
- tangent space functor
- Tangent bundle TM of a smooth manifold M
- sett.Man.T.tensor
- Tangent frames on smooth manifolds

tangent space functor

Definition. tangent space functor

Define the functor

$$T_* \colon \mathsf{Man}_* \to \mathsf{Vec}_{\mathbb{R}}$$

such that

- $$(M,p) \mapsto T_pM$$
- Smooth functions between the smooth manifolds maps onto its derivative at the basepoint

$$(F : (M,p) \to (N, F(p)) \mapsto \mathfrak{D}_pF$$

Tangent space functor is a covariant functor.

Current note has 8 direct children and 12 total descendants.

- structures based on sets
  - manifolds
    - smooth $\mathbb{R}$-manifold
      - Tangents on a smooth manifold
        - Tangent space of a smooth manifold
          - Canonical bijection of $T_p M$ with equivalence classes of velocities
          - Tangents on smooth manifolds as derivations $\mathcal{C}^\infty(M) \to \mathbb{R}$
          - local basis of a tangent space of a smooth manifold
          - Tangent bundle TM of a smooth manifold M
          - Cotangent bundle T^*M on a smooth manifold M
          - Comparison of constructions with tangent vectors
          - Tangent frames on smooth manifolds
          - Tangent vector fields on a smooth manifold
            - Vector field on smooth manifold M as derivation of $\mathcal{C}^\infty(M)$
            - Exponential of vector fields
            - Index of a vector field
            - Lie bracket of two smooth vector fields on manifold

And it has 4 siblings.

- structures based on sets
  - manifolds
    - smooth $\mathbb{R}$-manifold
      - Tangents on a smooth manifold
        - Cotangent space on a smooth manifold T_p^*M
        - Connection on the tangent bundle of a smooth manifold
        - Differential forms on smooth manifolds
        - Tangent space of a smooth manifold