

Info

This note, that [was found here](#), as a part of [a collection](#) is written (completely with human hands) by [Rupadarshi Ray](#), was created on February 16, 2026 8:09:24 PM, was last modified on August 28, 2026 3:09:48 PM, and was exported on September 7, 2026 3:25:05 PM.

Isometric action on a metric space

Definition. Distance on the quotient space of isometric action on a metric space

Let (X, d_X) is a metric space and $\Gamma \leq \text{Isom}(X)$ be a group acting by isometries on X . Then the quotient space

$$X \rightarrow \Gamma \backslash X$$

has the distance

$$d_\Gamma : \Gamma \backslash X \times \Gamma \backslash X \rightarrow [0, \infty]$$

$$d_\Gamma(\Gamma \{x\}, \Gamma \{y\}) := \inf \{d(\gamma x, \gamma y) | x, y \in X\}$$

Proposition:

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is a metric $\iff$ each Γ -orbit of X is a closed set.

In this case, the d_Γ -topology agrees with the quotient topology: for each $x \in X, r > 0$ we have

$$\Gamma B_r^{d_X}(x) = B_r^{d_\Gamma}(\Gamma \{x\})$$

fundamental domains

Definition. Fundamental domain of an isometric action

Let (X, d_X) is a metric space. A closed subset $F \subseteq X$ is a **(closed) fundamental domain** for $\Gamma \leq \text{Isom}(X)$ if

- $X = \bigcup_{\gamma \in \Gamma} \gamma(F) \iff F \rightarrow \Gamma \backslash X$ is surjective
- $\{\gamma(\mathring{F}) | \gamma \in \Gamma\}$ is a pairwise disjoint collection of open sets $\iff \mathring{F} \hookrightarrow \Gamma \backslash X$ is injective

Proposition: A fundamental domain exists $\implies$ orbits are discrete.

Definition. Locally finite fundamental domains

Let (X, d_X) is a metric space. A (closed) fundamental domain $F \subseteq X$ for $\Gamma \leq \text{Isom}(X)$ is **locally finite** if $\{\gamma(F) | \gamma \in \Gamma\}$ is a locally finite collection of closed sets.

Example Let (X, d_X) is a metric space and F be a fundamental domain for $\Gamma \leq \text{Isom}(X)$. Consider the commutative diagram

$$\begin{array}{ccc} F & \xhookrightarrow{\iota} & X \\ \downarrow \pi' & & \downarrow \pi \\ F/\Gamma & \xrightarrow{\bar{\iota}} & X/\Gamma \end{array}$$

The continuous bijection $\bar{\iota}$ is a homeomorphism $\iff F$ is a locally finite fundamental domain.

Lemma Let F be a locally finite fundamental domain.

- Let $U \subseteq F/\Gamma$ be open. As π' is continuous $\pi'^{-1}(U) \subseteq F$ is open which means there is a open $V \subseteq X$ such that

$$\pi'^{-1}(U) = F \cap V$$

- Consider

$$W := \bigcup_{\gamma \in \Gamma} \gamma(F \cap V)$$

$$\implies \pi(W) = \pi(F \cap V)$$

$$= \bar{\iota} \pi'(F \cap V)$$

$$= \bar{\iota}(U)$$

Intuition

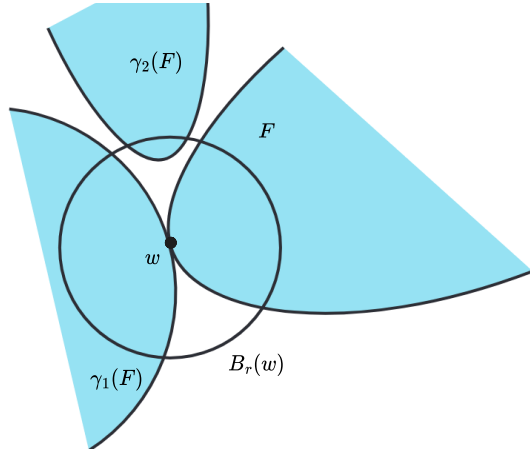
We must show W is open ($\implies \pi(W)$ is open as π is open $\implies \bar{\iota}$ is an open map $\implies \bar{\iota}$ is a homeomorphism). However

$$\bar{\iota}|_{\dot{F}/\Gamma} = \underbrace{\pi}_{\text{open}} \circ \underbrace{\iota|_{\dot{F}}}_{\text{open}} \circ \underbrace{\pi'^{-1}|_{\dot{F}/\Gamma}}_{\text{open}}$$

is therefore open. Hence, we must check open-ness on the boundary ∂F

- Let $\gamma w \in W$ be arbitrary where $w \in F \cap V$. As F is locally finite, there is a $r_1 > 0$ such that $B_r(w)$ for $r > r_1$ intersects only finitely many $\gamma(F)$, say, $\gamma_1(F), \dots, \gamma_m(F)$, so we have

$$r > r_1 \implies B_r(w) \subseteq \gamma_1(F) \cup \dots \cup \gamma_m(F)$$



- If $\gamma_i(F)$ does not contain w then

$$\underbrace{B_r(w) \setminus \underbrace{\gamma_i(F)}_{\text{closed}}}_{\text{open}}$$

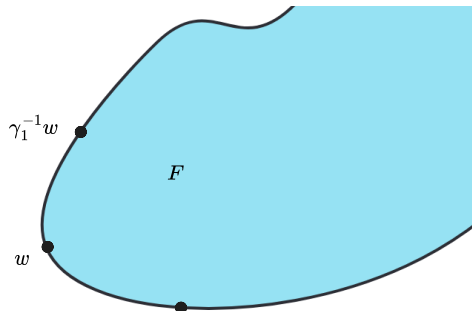
is a onb of w that does not intersect $\gamma_i(F)$. Therefore there is a ball at w inside $B_r(w) \setminus \gamma_i(F)$.

- By shrinking r_1 to a smaller $r_2 > 0$ we have

$$r > r_2 \implies B_r(w) \subset \bigcup_{w \in \gamma_i(F)} \gamma_i(F)$$

- Now

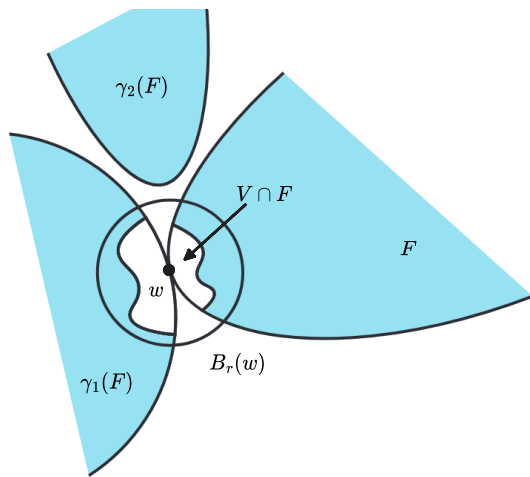
$$\begin{aligned} w &\in \gamma_i(F) \\ \implies \gamma_i^{-1}(w) &\in F \\ \implies \gamma_i^{-1}(w) &\in F \cap V \quad \text{as } \pi'(\gamma_i^{-1}(w)) = \pi'(w) \in \pi'(F \cap V) \\ \implies w &\in \gamma_i(F \cap V) \end{aligned}$$



- As

$$\bigcup_{w \in \gamma_i(F)} \underbrace{\gamma_i(V)}_{\text{open}}$$

is open, there is a ball near w inside it.



Thus, by shrinking r_2 still further to $r_3 > 0$ we may assume

$$r > r_3 \implies B_r(w) \subseteq \bigcup_{w \in \gamma_i(F)} \gamma_i(V)$$

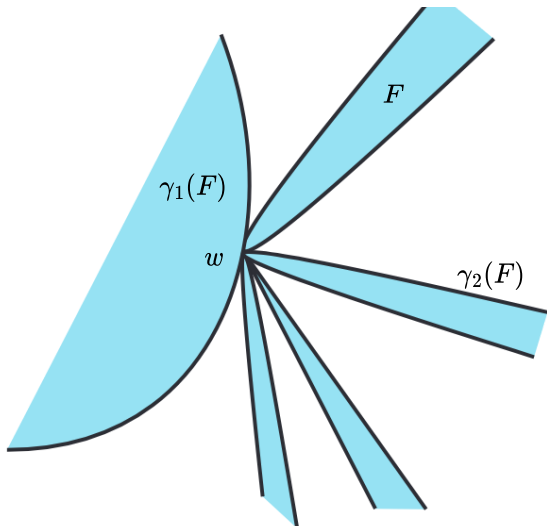
- Therefore,

$$\begin{aligned} r > r_3 \implies B_r(w) &\subseteq \bigcup_{w \in \gamma_i(F)} \gamma_i(V) \cap \bigcup_{w \in \gamma_i(F)} \gamma_i(F) \\ &\subseteq \bigcup_{w \in \gamma_i(F)} \gamma_i(V \cap F) \\ &\subseteq W \end{aligned}$$

so W is open.

💡 Let F is **not** locally finite, that is, there is a $w \in F$ such that

$$\forall \epsilon > 0, \left| \underbrace{\{\gamma \in \Gamma | B_\epsilon(w) \cap \gamma F \neq \emptyset\}}_{F_\epsilon(w)} \right| = \infty$$



Then

$$\gamma_n \in \left\{ \gamma \in \Gamma \mid B_{\frac{1}{n}}(w) \cap \gamma F \neq \emptyset \right\} \setminus \{ \gamma_1, \dots, \gamma_{n-1} \}$$

- 
Intuition

We must show $\bar{\iota}$ is **not** closed.

- Thus

$$\underbrace{\gamma_n w_n}_{\in F} \xrightarrow{n \rightarrow \infty} w \in F$$

and for $K := \{w_n\}$ but

$$\pi(\gamma_n w_n) = \underbrace{\pi(w_n)}_{\in \pi(K)} \xrightarrow{n \rightarrow \infty} \pi(w) \notin \pi(K)$$

So $\pi(K)$ is **not** closed.

- Choose

$$r := \frac{1}{2} d(x, \Gamma \{y\} \setminus \{x\}) > 0$$

- As γ_n are all distinct and $x = \gamma_n^{-1} w$ for $\gamma_n \in \Gamma_w$ which is finite. Thus

$$d(w, \gamma_n^{-1} w) \geq 2r$$

- Thus $B_r(x) \cap K$ is finite so there is an open ball at x avoiding K , thus K is closed.
- Now

$$\pi'^{-1}(\pi'(K)) = \underbrace{K}_{\text{closed}}$$

thus $\pi'(K)$ is closed.

- However

$$\underbrace{\pi'(K)}_{\text{closed}} \xrightarrow{\bar{\iota}} \underbrace{\pi(K)}_{\text{not closed}}$$

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- [structures based on sets](#)
 - [Metric spaces](#)
 - [Metric space isometries](#)
 - [Isometric action on a metric space](#)
 - [Dirichlet domain of group acting isometrically and finitely discontinuously.](#)

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 - [Metric spaces](#)
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