Info

This note, that was found here, as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
was created on January 24, 2026 10:37:57 PM,
was last modified on January 25, 2026 12:10:12 AM, and
was exported on September 7, 2026 3:04:07 PM.

$\mathrm{AutPoly}(\mathbb{R}^n)$

Definition.

$$\mathrm{AutPoly}(\mathbb{R}^n) := \left\{ f : \mathbb{R}^n \to \mathbb{R}^n \middle| f, f^{-1} \in \mathbb{R}[\mathbb{R}^n] \right\}$$

compact group actions by polynomial automorphisms have Zariski closed orbits

> **Proposition:** Let G be a compact group and $\theta : G \curvearrowright \mathbb{R}^m$ be a smooth action for some $m \geq 1$ such that G acts by polynomial automorphisms
>
> $$\theta : G \to \mathrm{AutPoly}(\mathbb{R}^n)$$
>
> Then every orbit
>
> $$G\{p\}$$
>
> (for any $p \in \mathbb{R}^n$) is a Zariski closed subset of $\mathbb{R}^n$.

Let $y, x \in \mathbb{R}^n$ be in different orbits. As $G\{x\}, G\{y\}$ are compact and disjoint, there is a continuous function

$$\begin{aligned} \varphi : \mathbb{R}^n &\to \mathbb{R} \\ G\{x\} &\mapsto 0 \\ G\{y\} &\mapsto 1 \end{aligned}$$

- By Stone-Weirstrass theorem there is a polynomial $p \in \mathbb{R}[\mathbb{R}^n]$ such that

$$\|\varphi - p\|_{\mathcal{C}(G\{x,y\})} < \frac{1}{4}$$

- So

$$\begin{aligned} |p(z)| &< \frac{1}{4} \text{ on } z \in G\{x\} \\ |p(z) - 1| &< \frac{1}{4} \text{ on } z \in G\{y\} \end{aligned}$$

- Now we average out this polynomial along every orbit

$$\begin{aligned} f : \mathbb{R}^m &\to \mathbb{R} \\ f(z) &:= \int_{g \in G} p(\theta^g(z)) \mathrm{d}\nu_G \end{aligned}$$

- As θ^g is a polynomial automorphism, we have

$$\begin{aligned} \theta^g(z) &= \sum_I c_I(g) z^I \\ p(z) &= \sum_I p_I z^I \\ \implies p(\theta^g(z)) &= \sum_I a_I(g) z^I \\ \implies f(z) = \int_{g \in G} p(\theta^g(z)) \mathrm{d}\nu_G &= \sum_I \left(\int_{g \in G} a_I(g) \mathrm{d}\nu_G \right) z^I \end{aligned}$$

  which is still a polynomial with

$$|f(x)| < \frac{1}{4}, |f(y) - 1| < \frac{1}{4}$$

- Therefore, $f - f(x)$ is a polynomial which vanishes on $G\{x\}$ and not on y.
- So the Zariski closure of $G\{x\}$ contains no element $y \in \mathbb{R}^n \setminus G\{x\}$. Thus the orbit $G\{x\}$ is Zariski closed.

Current note has 0 direct children and 0 total descendants.

- stretchy
  - $\mathbb{R}^n$
    - Automorphisms of $\mathbb{R}^n$
      - $\mathrm{AutPoly}(\mathbb{R}^n)$

And it has 1 siblings.

- stretchy
  - $\mathbb{R}^n$
    - Automorphisms of $\mathbb{R}^n$
      - $\mathrm{AutPoly}(\mathbb{R}^n)$