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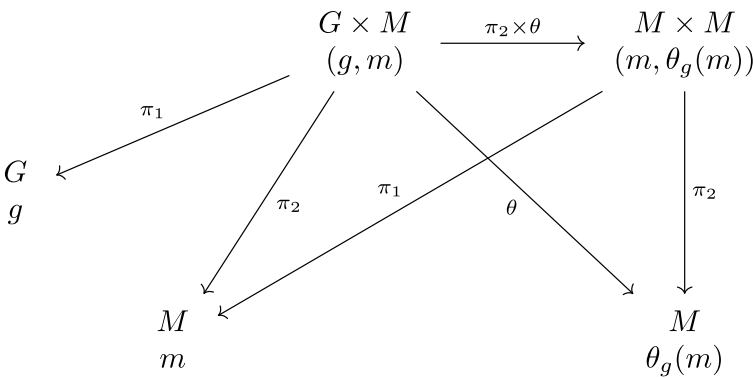
Proper Lie group actions

Definition. A smooth Lie group action $\theta : G \curvearrowright M$ is **proper** if

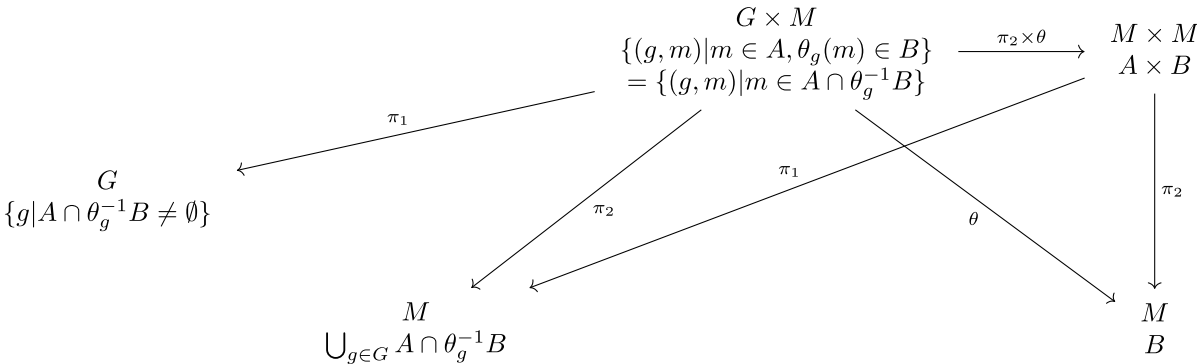
$$\begin{aligned} \pi_2 \times \theta : G \times M &\rightarrow M \times M \\ (g, m) &\mapsto (m, \theta_g(m)) \end{aligned}$$

is *proper* as a continuous map.

We have a following commutative diagram



If $A, B \subseteq M$ be subsets then



equivalent characterizations

Proposition: Let M be a topological manifold and let $G \curvearrowright M$ acts continuously. Then the following are equivalent

- the action is proper

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- $\{p_i\} \subseteq M, \{g_i\} \subseteq G$ are sequences such that both $\{p_i\}, \{g_i p_i\}$ converge $\implies$ a subsequence of $\{g_i\}$ converges
- for every compact $K \subseteq M$ the set

$$\begin{aligned} G_K &:= \{g \in G | g(K) \cap K \neq \emptyset\} \\ &:= \pi_{G \times M \rightarrow G}((\pi_2 \times \theta)^{-1}(K \times K)) \end{aligned}$$

is compact.

proper actions and compactness

Proposition: Any smooth Lie group action by a *compact* Lie group is proper.

Proposition: Let $G \curvearrowright M$ be a smooth and proper Lie group action on a compact manifold M . Then G must be **compact**.

restricted action by closed subgroups

Let $G \curvearrowright X$ is a proper action and $H \leq G$ is a closed subgroup. Then

$$H \curvearrowright X := H \hookrightarrow G \curvearrowright X$$

is a **proper action**.



$$H \times X \xrightarrow{\text{proper}} G \times X \xrightarrow{\text{proper}} X \times X$$

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- [structures based on sets](#)
 - [manifolds](#)
 - [smooth \$\mathbb{R}\$ -manifold](#)
 - [Smooth Lie group action on a smooth manifold](#)
 - [Proper Lie group actions](#)

And it has 5 siblings.

- [structures based on sets](#)
 - [manifolds](#)
 - [smooth \$\mathbb{R}\$ -manifold](#)

- Smooth Lie group action on a smooth manifold
 - Smooth action by a compact Lie group
 - Generators of a Lie group action
 - Orbits of smooth Lie group actions
 - Proper Lie group actions
 - invariant Riemannian metric in a proper Lie group action