> ⓘ Info
>
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Proper Lie group actions

> **Definition.** A smooth Lie group action $\theta : G \curvearrowright M$ is **proper** if
>
> $$\begin{aligned} \pi_2 \times \theta : G \times M &\to M \times M \\ (g, m) &\mapsto (m, \theta_g(m)) \end{aligned}$$
>
> is *proper* as a continuous map.

We have a following commutative diagram

$$\begin{tikzcd}
& \begin{array}{c} G \times M \\ (g,m) \end{array} \arrow[dl, "\pi_1"'] \arrow[d, "\pi_2"] \arrow[drr, "\theta"] \arrow[rr, "\pi_2 \times \theta"] & & \begin{array}{c} M \times M \\ (m, \theta_g(m)) \end{array} \arrow[dll, "\pi_1"'] \arrow[d, "\pi_2"] \\
\begin{array}{c} G \\ g \end{array} & \begin{array}{c} M \\ m \end{array} & & \begin{array}{c} M \\ \theta_g(m) \end{array}
\end{tikzcd}$$

If $A, B \subseteq M$ be subsets then

$$\begin{tikzcd}
& & \begin{array}{c} G \times M \\ \{(g,m) | m \in A, \theta_g(m) \in B\} \\ = \{(g,m) | m \in A \cap \theta_g^{-1} B\} \end{array} \arrow[dll, "\pi_1"'] \arrow[dl, "\pi_2"] \arrow[dr, "\theta"] \arrow[r, "\pi_2 \times \theta"] & \begin{array}{c} M \times M \\ A \times B \end{array} \arrow[dll, "\pi_1"'] \arrow[d, "\pi_2"] \\
\begin{array}{c} G \\ \{g | A \cap \theta_g^{-1} B \neq \emptyset\} \end{array} & \begin{array}{c} M \\ \bigcup_{g \in G} A \cap \theta_g^{-1} B \end{array} & & \begin{array}{c} M \\ B \end{array}
\end{tikzcd}$$

equivalent characterizations

> **Proposition:** Let M be a topological manifold and let $G \curvearrowright M$ acts continuously. Then the following are equivalent
>
> 1. the action is proper
>
>   > **Definition.** A smooth Lie group action $\theta : G \curvearrowright M$ is **proper** if
>   >
>   > $$\begin{aligned} \pi_2 \times \theta : G \times M &\to M \times M \\ (g, m) &\mapsto (m, \theta_g(m)) \end{aligned}$$
>   >
>   > is *proper* as a continuous map.
>
> 2. $\{p_i\} \subseteq M, \{g_i\} \subseteq G$ are sequences such that both $\{p_i\}, \{g_i p_i\}$ converge $\implies$ a subsequence of $\{g_i\}$ converges
> 3. for every compact $K \subseteq M$ the set
>
> $$\begin{aligned} G_K &:= \{g \in G | g(K) \cap K \neq \emptyset\} \\ &:= \pi_{G \times M \to G}((\pi_2 \times \theta)^{-1}(K \times K)) \end{aligned}$$
>
>   is compact.

proper actions and compactness

> **Proposition:** Any smooth Lie group action by a *compact* Lie group is proper.

> **Proposition:** Let $G \curvearrowright M$ be a smooth and proper Lie group action on a compact manifold M. Then G must be **compact**.

restricted action by closed subgroups

> Let $G \curvearrowright X$ is a proper action and $H \leq G$ is a closed subgroup. Then
>
> $$H \curvearrowright X := H \hookrightarrow G \curvearrowright X$$
>
> is a **proper action**.

$$H \times X \underbrace{\to}_{\text{proper}} G \times X \underbrace{\to}_{\text{proper}} X \times X$$

Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - manifolds
    - smooth $\mathbb{R}$-manifold
      - Smooth Lie group action on a smooth manifold
        - Proper Lie group actions

And it has 5 siblings.

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