> ⓘ Info
>
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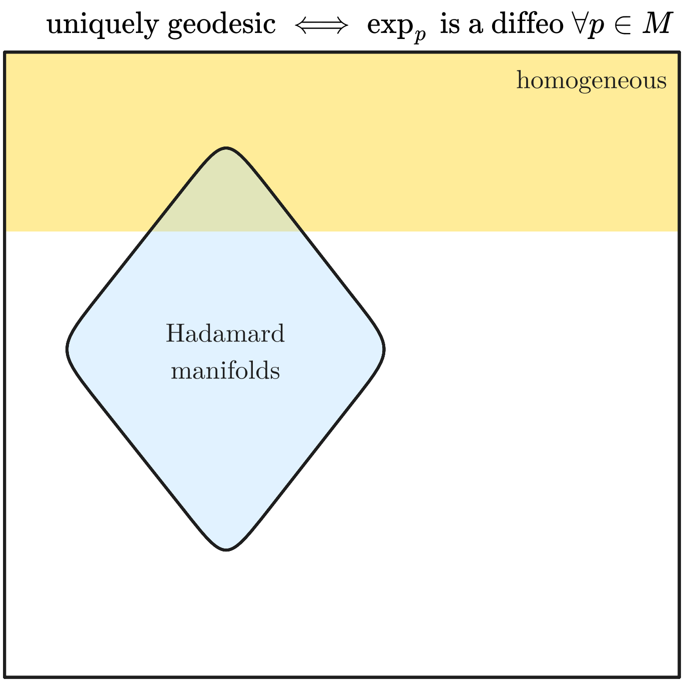
Uniquely geodesic complete Riemannian manifolds

> **Proposition:** Let (M, g) be a complete Riemannian manifold. Then the following are equivalent
>
> 1. every geodesic in (M, g) is (globally) **length-minimizing**
> 2. every unit speed geodesic in (M, g) is a minimal geodesic
> 3. every pair of points in (M, g) is joined by a **unique** minimal geodesic segment
> 4. the exponential
>
> $$\exp_p : T_pM \to M$$
>
> is a diffeomorphism for all $p \in M$

[1]



embedded *completely* geodesic submanifolds

> **Definition.** Embedded *completely* geodesic submanifolds
>
> Let $S \subseteq M$ be a connected, *embedded* submanifold of a uniquely geodesic Riemannian manifold (M, g). Then S is called **completely geodesic** if for every g-geodesic $\gamma : \mathbb{R} \to M$, if γ meets S in two points then this implies $\gamma(\mathbb{R}) \subset \iota(S)$.

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> 1. **Definition.** Embedded *completely* geodesic submanifolds
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> 2. $$\forall q \in S, \exp^{\mathrm{g}}_q(T_qS) \subseteq \iota(S)$$
>
> In any (hence all) of these cases
>
> - for any $p \in S$, $\exp^{(M,\mathrm{g})}_p(T_pS) = S$
> - $d_{\iota^*\mathrm{g}} = \iota^* d_{\mathrm{g}}$
> - $(S, \iota^*\mathrm{g})$ is a complete Riemannian manifold

Assume (1). Then

$$d_{\iota^*\mathrm{g}} = \iota^* d_{\mathrm{g}}$$

- Thus a g-geodesic must agree with ι^*g-geodesic.
- Thus (2) follows.

Assume (2). Let $p, q \in S$. Then p, q

- By

  > **Proposition:**
  > Let
  >
  > $$\iota : (M, \iota^*\mathrm{g}) \hookrightarrow (X, \mathrm{g})$$
  >
  > be an *embedded* Riemannian submanifold. Then the following is equivalent
  >
  > - M is totally geodesic
  >
  >   > **Definition.** Totally geodesic submanifolds of semi-Riemannian manifolds
  >   >
  >   > A Riemannian (immersed/embedded) submanifold
  >   >
  >   > $$\iota : (M, \mathrm{g}_M) \hookrightarrow (X, \mathrm{g}_X)$$
  >   >
  >   > of a semi-Riemannian manifold (X, g_X) is **totally geodesic** if for every g_X-geodesic that is tangent to M at some $t_0 \in \mathbb{R}$ stays in M for all $t \in (t_0 - \epsilon, t_0 + \epsilon)$.
  > - Every ι^*g-geodesic $\gamma : (a, b) \to M$ is also a g-geodesic $\iota \circ \gamma : (a, b) \to X$
  > - The second fundamental form of M vanishes identically.

> we have ...

Current note has 0 direct children and 0 total descendants.

- structures based on sets
    - manifolds
        - $\mathbb{R}$-manifold semi-Riemannian
            - Complete Riemannian manifolds
                - Uniquely geodesic complete Riemannian manifolds

And it has 14 siblings.

- structures based on sets
    - manifolds
        - $\mathbb{R}$-manifold semi-Riemannian
            - Complete Riemannian manifolds
                - Integrating sectional curvature
                - Riemannian manifolds with negative curvature
                - Riemannian manifolds with non-positive sectional curvature
                - Cut locus in a complete Riemannian manifold
                - Exponential map for a complete Riemannian manifold
                - Geodesic completeness $\iff$ metric completeness of Riemannian manifolds
                - Geodesics in homotopy classes
                - Homogeneous Riemannian manifolds
                - Homogeneous and isotropic Riemannian manifolds
                - Poles of a complete Riemannian manifold
                - Riemannian symmetric spaces
                - Uniquely geodesic complete Riemannian manifolds
                - Riemannian homogeneous manifolds where $\exp_p$ is a diffeomorphism
                - Complete Riemannian manifolds without conjugate points

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1. https://math.stackexchange.com/a/1530561/1290493 ↩