ⓘ Info

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Factorization of holomorphic functions on $\mathbb{C}$

infinite products

Definition. *Non-zero* product of a sequence of non-zero complex numbers

Given a sequence $(p_n) \subset \mathbb{C} \setminus \{0\}$ we have the partial products

$$p_0, p_0p_1, \ldots, \prod_{n \le k} p_n, \ldots$$

Then the **infinite product** is defined to be the limit of the sequence of partial products

$$\prod_{n \ge 0} p_n := \lim_{k \to \infty} \prod_{n \le k} p_n$$

only if the right side is not 0.

- Let $p_n \in \mathbb{C} \setminus \{0\}$ be such that

$$\prod_{n \ge 0} p_n = P \neq 0$$

  then

$$p_n = \frac{\prod_{n \le k} p_n}{\prod_{n \le k-1} p_n} \xrightarrow{k \to \infty} 1$$

  - Thus it is necessary

$$p_n \xrightarrow{n \to \infty} 1$$

    for the infinite product of p_n to converge.

$\prod_{n \ge 1} p_n$ converges

$$\iff$$

$\sum_n \log(p_n)$ converges.

constructing an entire function with given zeros and multiplicities

We shall use *Weierstrass' primary factors* and their bound

bound on $1 - E_n$

Proposition: The functions

Definition. Weirstrass' primary factors

$$E_0(z) := 1 - z$$
$$E_n(z) := (1 - z) \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^n}{n}\right)$$

for $n \in \mathbb{Z}_{>0}$

satisfy

$$|1 - E_n(z)| \le |z|^{n+1}$$

on the unit disk D for all n.

For $m = 0$

$$|1 - (1 - z)| = |z|^1$$

For $m \ge 1$ we have

$$E_m'(z) = -z^m \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^m}{m}\right)$$

which means

$$(1 - E_m)'(z) = z^m \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^m}{m}\right)$$

.

- As

$$\frac{1 - E_m(z)}{z^{m+1}} = \sum_{n \ge 0} a_n z^n$$
$$1 - E_m(z) = \sum_{n \ge 0} a_n z^{n+m+1}$$

  we have

$$\sum_{n \ge 0} \frac{a_n}{n + m + 1} z^{n+m} = -z^m \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^m}{m}\right)$$
$$= -z^m \sum_{n \ge 0} \frac{1}{n!} \left(z + \frac{z^2}{2} + \cdots + \frac{z^m}{m}\right)^n$$

  Thus

$$a_n \ge 0$$

- So

$$(1-E_m)(0)=0$$

and $1-E_m$ has a zero of order $m+1$ at $z=0$. Thus

$$\frac{1-E_m(z)}{z^{m+1}}$$

has a removable singularity on 0, so it extends to an entire function

$$\begin{aligned}\frac{1-E_m(z)}{z^{m+1}} &= \sum_{n\geq 0} a_n z^n \\ \left|\frac{1-E_m(z)}{z^{m+1}}\right| &\leq \sum_{n\geq 0}|a_n||z|^n \leq \sum_{n\geq 0}|a_n| \text{ on } D \\ &= \sum_{n\geq 0} a_n \\ &= \frac{1-E_m(1)}{1^{m+1}} = 1\end{aligned}$$

- Therefore,

$$|z|\leq 1 \implies |1-E_m(z)|M = |z|^{m+1}$$

(Entire function with given zeros and multiplicities) For a sequence $(a_n)\subset\mathbb{C}$ such that

$$|a_n| \xrightarrow{n\to\infty} \infty$$

let m be the number of times 0 occurs in a_n. Then

$$z^m \prod_{a_n\neq 0} E_n\left(\frac{z}{a_n}\right) : \mathbb{C}\to\mathbb{C}$$

converges to a *holomorphic function* (entire) that has zeroes precisely at $\{a_n\}\subset\mathbb{C}$ with multiplicity at a zero a equal to the cardinality $|\{n \mid a_n = a\}|$.

- On choosing ϕ_n suitably such that

$$z^m \prod_{a_n\neq 0} E_{\phi_n}\left(\frac{z}{a_n}\right)$$

on a disk of radius $R>0$

$$\sum_{n>N}\left|1-E_{\phi_n}\left(\frac{z}{a_n}\right)\right| \leq \sum_{n>N}\left|\frac{z}{a_n}\right|^{\phi_n+1} \leq \sum_{n>N}\left|\frac{R}{a_n}\right|^{\phi_n+1}$$

converges.

Weierstrass Factorization Theorem

Let f be an entire function, $f\not\equiv 0$, and let $k\geq 0$ be the order of the zero of f at 0. Let the remaining zeros of f be at $z_1, z_2, \ldots$, where each z_n is repeated as often as its multiplicity. Then

$$f(z) = e^{g(z)} z^k \prod_n E_{m_n}(z/z_n)$$

for some entire function g and nonnegative integers m_n.

Proof. If f has finitely many zeros, the result is immediate, so assume that there are infinitely many z_n. Since $f\not\equiv 0, |z_n|\to\infty$. By (6.2.3) there is a sequence $\{m_n\}$ such that

$$h(z) = f(z)/[z^k \prod_{n=1}^{\infty} E_{m_n}(z/z_n)]$$

has a zero-free extension to an entire function, which we will persist in calling h. But now h has an analytic logarithm g on $\mathbb{C}$, hence $h(z)=e^{g(z)}$ and we have the desired representation. ♣

More generally, versions of (6.2.3) and its consequence (6.2.5) are available for any *proper* open subset of $\hat{\mathbb{C}}$. We begin with the generalization of (6.2.3).

Holomorphic function with prescribed zeros and their multiplicities

(Holomorphic function on open subsets of $\mathbb{C}P^1$ with given zeros and their multiplicities) Let Ω be a proper open subset of $\mathbb{C}P^1$ and $A\subseteq\Omega$ with no limit point in Ω. For each $a\in A$, let $m_a\in\mathbb{Z}_{>0}$ be a positive integer. Then there exists a holomorphic function

$$f:\Omega\to\mathbb{C}$$

such that $A=f^{-1}(0)$ and for each $a\in A$ the multiplicity of f at a is m_a.

Ash and Novinger - Complex Variables , p.144

Current note has 0 direct children and 0 total descendants.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, _\Gamma\backslash^{\mathbb{C}^n}$
    - Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - Factorization of holomorphic functions on $\mathbb{C}$

And it has 19 siblings.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, _\Gamma\backslash^{\mathbb{C}^n}$
    - Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - Approximation of holomorphic functions on $\mathbb{C}$ by rational functions

- Embedding unit disk D into $\mathbb{C}^3$
- Factorization of holomorphic functions on $\mathbb{C}$
- Global holomorphic functions
- Equivalent descriptions of holomorphic functions
- Meromorphic functions on the complex plane
- Holomorphic functions meromorphic at infinity
- Modular forms
- Reconstructing meromorphic functions from stalks
- Extending holomorphic functions by reflections
- Rotational symmetrization of holomorphic functions
- Sheaf of holomorphic functions on $\mathbb{C}$
- $\mathscr{O}(U)$
- $\mathscr{O}(\mathbb{C})$
- $\mathscr{O}(D)$
- $\mathscr{O}^p(H^2_{\mathrm{U}})$
- $\mathscr{O} \cap L^p$
- $\mathscr{O}(S^1)$
- Zeros and singularities of holomorphic functions