ⓘ Info

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Quasiconformal maps on $\mathbb{C}$

Definition. Quasiconformal maps between open subsets of $\mathbb{C}$

Let $U, V \subseteq \mathbb{C}$ and $K \geq 1$. Then a homeomorphism

$$f : U \to V$$

is K-**quasiconformal** if its distributional partial derivatives are in $L^2_{\mathrm{loc}}(U)$ and satisfy

$$\left| \frac{\overset{\int}{\partial} f}{\partial \overline{z}} \right| \leq \frac{K-1}{K+1} \left| \frac{\overset{\int}{\partial} f}{\partial z} \right| \text{ aeon } U$$

- The set of all such maps

$$\mathscr{Q}_K(U, V) \subseteq H^1_{\mathrm{loc}} \cap \mathcal{C}(U)$$

is contained in the set of all quasiconformal maps

$$\mathscr{Q}(U, V) := \bigcup_{K \geq 1} \mathscr{Q}_K(U, V)$$

- We also have the set of all quasiconformal mapping into $\mathbb{C}$ $\mathscr{Q}(U) := \bigcup_{V \subseteq \mathbb{C}} \mathscr{Q}(U, V)$ and $\mathscr{Q}_K(U) := \bigcup_{V \subseteq \mathbb{C}} \mathscr{Q}_K(U, V)$.
- We have the **Beltrami coefficient** of a quasiconfomal map

$$\begin{aligned} \mu : \mathscr{Q}(U) &\to L^\infty(U) \\ f &\mapsto \mu(f)(z) := \frac{\overset{\int}{\partial} f}{\partial \overline{z}} \left(\frac{\overset{\int}{\partial} f}{\partial z} \right)^{-1} \end{aligned}$$

and the **eccentricity**

$$\begin{aligned} K : \mathscr{Q}(U) &\to L^\infty(U) \\ f &\mapsto K(f)(z) := \frac{1 + |\mu(f)(z)|}{1 - |\mu(f)(z)|} \end{aligned}$$

The smallest K such that f is K-quasiconformal is called the **quasiconformal constant** $\|K(f)\|_\infty$ of f.

(Weyl's lemma)

$$\mathscr{Q}_1(U, V) \subset \mathscr{O}(U, V)$$

The map

$$\begin{aligned} \mu : \mathscr{Q}(U) &\to L^\infty(U) \\ f &\mapsto \mu(f)(z) := \frac{\overset{\int}{\partial} f}{\partial \overline{z}} \left(\frac{\overset{\int}{\partial} f}{\partial z} \right)^{-1} \end{aligned}$$

is **surjective** and the fiber containing $f \in \mathscr{Q}(U)$ is precisely left composition of f by injective holomorphic mappings

$$\mu^{-1}(\mu(f)) = \{\varphi \circ f | \varphi \in \mathscr{Q}_1(f(U))\}$$

[1]

extension to boundary

$$\mathscr{O}_K(H^2_{\mathrm{U}}, H^2_{\mathrm{U}}) \hookrightarrow \mathcal{C}(\mathbb{R}\boldsymbol{P}^1, \mathbb{R}\boldsymbol{P}^1)$$

and

$$\begin{aligned} \mathscr{O}_K(H^2_{\mathrm{U}}, H^2_{\mathrm{U}}) &\hookrightarrow \mathscr{O}_K(\mathbb{C}\boldsymbol{P}^1, \mathbb{C}\boldsymbol{P}^1) \\ f &\mapsto \begin{cases} f \\ f(\overline{z}) = \overline{f(z)} \end{cases} \end{aligned}$$

Current note has 0 direct children and 0 total descendants.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, {}_\Gamma\backslash^{\mathbb{C}^n}$
    - quasiconformal
      - Quasiconformal maps on $\mathbb{C}$

And it has 2 siblings.

- stem
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      - Quasiconformal maps on S^n

Theorem 4.6.1 (The mapping theorem)

1. *Let $U \subset \mathbb{C}$ be open and let $\mu \in L^\infty(U)$ satisfy $\|\mu\|_\infty < 1$. Then there exists a quasiconformal mapping $f : U \to \mathbb{C}$ satisfying the Beltrami equation*

$$\frac{\partial f}{\partial \bar{z}} = \mu \frac{\partial f}{\partial z}. \tag{4.6.1}$$

2. *If g is another quasiconformal solution to equation 4.6.1, then there exists an injective analytic function $\varphi : f(U) \to \mathbb{C}$ such that $g = \varphi \circ f$.*

1. ↩