Info

This note, that was found here, as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
was created on March 8, 2026 5:00:14 PM,
was last modified on August 28, 2026 5:00:59 PM, and
was exported on September 7, 2026 3:09:47 PM.

Elementary Fuchsian groups

> **Proposition:** Let $g \in PSL(2)(\mathbb{R})$ be parabolic and $h \in PSL(2)(\mathbb{R})$ be such that $gh = hg$. Then
>
> - $g\xi = \xi \implies h\xi = \xi$
> - if $\langle g, h \rangle$ is discrete then $\langle g, h \rangle \cong \mathbb{Z}$.

- Now we may conjugate g, h and assume $\xi = \infty$.
- As g is parabolic

$$g(z) = z + b, h(z) = cz + d$$

- Now,

$$\begin{aligned} gh &= hg \\ \implies a(cz+d)+b &= c(az+b)+d \\ \implies ad+b &= cb+d \\ \implies d(a-1) &= b(c-1) \end{aligned}$$

- $gh = hg \implies g, h$ are translations
- Thus if they generate a discrete subgroup, it must be a discrete subgroup of $\mathbb{R}$.
- Hence, $\langle g, h \rangle \cong \mathbb{Z}$

> **Proposition:** Let $g \in PSL(2)(\mathbb{R})$ be hyperbolic and $h \in PSL(2)(\mathbb{R})$ be such that $gh = hg$. Then
>
> - $\text{fix}(g) = \text{fix}(h)$
> - if $\langle g, h \rangle$ is discrete then $\langle g, h \rangle \cong \mathbb{Z}$

No Fuchsian group is isomorphic to $\mathbb{Z}^2$.

Assume we have $\Gamma = \langle g, h \rangle \cong \mathbb{Z}^2$. That means g generates a discrete subgroup with $\langle g \rangle \cong \mathbb{Z}$

- If g is elliptic, then g must have finite order which
- If g is hyperbolic,

  > **Proposition:** Let $g \in PSL(2)(\mathbb{R})$ be hyperbolic and $h \in PSL(2)(\mathbb{R})$ be such that $gh = hg$. Then
  >
  > - $\text{fix}(g) = \text{fix}(h)$
  > - if $\langle g, h \rangle$ is discrete then $\langle g, h \rangle \cong \mathbb{Z}$

  we have $\Gamma \cong \mathbb{Z}^2$
- If g is parabolic,

  > **Proposition:** Let $g \in PSL(2)(\mathbb{R})$ be parabolic and $h \in PSL(2)(\mathbb{R})$ be such that $gh = hg$. Then
  >
  > - $g\xi = \xi \implies h\xi = \xi$
  > - if $\langle g, h \rangle$ is discrete then $\langle g, h \rangle \cong \mathbb{Z}$.

  we have

Let Γ be a Fuchsian group whose each element is elliptic. Then $\Gamma \leq PSL(2)(\mathbb{R})_p$ for some $p \in H^2_{\mathbb{U}}$ is finite cyclic.

There are two cases.

- Either $\Gamma \leq PSL(2)(\mathbb{R})_p \cong SO(2)(\mathbb{R})$
  - Then Γ is finite cyclic.
- There are g, h which fix different elements.
  - We may conjugate with $l \in PSL(2)(\mathbb{R})$ such that g, h fixes $i, \lambda i$ respectively.
  - Then $\text{tr}([g, h]) \geq 2$ which is a contradiction to the fact that every element of Γ is elliptic.

A Fuchsian group is elementary $\iff$ it is virtually cyclic.

($\implies$) Let Γ be elementary $\implies$ has a finite orbit O. Then Γ has a finite index subgroup

$$\Gamma' := \bigcap_{p \in O} \Gamma_p \leq \Gamma$$

fixing the orbit pointwise. So there are two cases.

- Γ' fixes $i \in H^2_{\mathbb{U}}$
  - Then $\Gamma' \leq SO(2)(\mathbb{R})$
- Γ' fixes ∞
  - if Γ' are all parabolic, then $\Gamma' \cong \mathbb{Z}$
  - if Γ' contains parabolic and hyperbolic

    $$g(z) = z + b, h(z) = cz + d$$

    then it is a contradiction
  - if Γ' are all hyperbolic, then it is cyclic

> **Proposition:**
>
> - $p \in H^2$ then $\Gamma \leq PSL(2)(\mathbb{R})_p$ and is finite cyclic
> - $p \in \partial H^2$ then
>   - $|\Gamma \{p\}| = 1 \implies \Gamma \cong \mathbb{Z}$

- $|\Gamma\{p\}| = 2$ implies there is a index 2 subgroup Γ' which fixes $0, \infty$ pointwise $\implies$ Γ' is either
  - trivial, so $\Gamma \cong \mathbb{Z}_2$
  - $\cong \langle\gamma\rangle$ generated by a hyperbolic isometry, then $\Gamma = \langle h, \gamma\rangle$ where $h(0) = \infty$, then Γ is conjugate to

$$\left\langle \begin{bmatrix} \lambda & \\ & \lambda^{-1} \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\rangle$$

- $|\Gamma\{p\}| > 2 \implies$ there is a finite index subgroup Γ' that fixes ≥ 3 points pointwise, so Γ is finite cyclic

Every Abelian Fuchsian group is cyclic.

Any elementary Fuchsian group is either cyclic or is conjugate in $PSL(2)(\mathbb{R})$ to a group generated by

$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}?$$

[1]

Current note has 0 direct children and 0 total descendants.

- stem
  - Objects stemming from $PSL(2)(\mathbb{R})$
    - Fuchsian groups $\Gamma <_d PSL(2)(\mathbb{R}) \leftrightarrow$ oriented hyperbolic 2-manifold/orbifolds $\to$ Riemann surfaces with branching signature
      - Elementary Fuchsian groups

And it has 17 siblings.

- stem
  - Objects stemming from $PSL(2)(\mathbb{R})$
    - Fuchsian groups $\Gamma <_d PSL(2)(\mathbb{R}) \leftrightarrow$ oriented hyperbolic 2-manifold/orbifolds $\to$ Riemann surfaces with branching signature
      - Artihmetic Fuchsian groups and Artihmetic hyperbolic 2-orbifolds
      - Cocompact Fuchsian groups
      - Central simple algebra
      - Elementary Fuchsian groups
      - Finitely generated Fuchsian groups $\leftrightarrow$ geometrically finite Fuchsian groups
      - Fuchsian groups of finite coarea
      - Geodesic flow of Fuchsian groups and hyperbolic surfaces
      - Decomposition of hyperbolic surfaces
      - Infinitely generated Fuchsian groups
      - Limit set of Fuchsian groups
      - Marked hyperbolic surfaces
      - subst.R PSL 2 discrete subg.over Q-bar
      - Fuchsian Schottky groups
      - Spectrum of hyperbolic surfaces
      - Torsion-free Fuchsian groups
      - Cocompact torsion-free Fuchsian groups
      - Fuchsian triangle groups

---

1. https://lib.ysu.am/open_books/417878.pdf ↩