Info

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Maximum and minimum of holomorphic functions

- Let $f: U(\text{open}) \subseteq \mathbb{C} \to \mathbb{C}$ be a non-constant holomorphic function, then by

  > **(Open mapping theorem for open subsets in $\mathbb{C}$)** Let f be *non-constant* holomorphic function on a connected open set U then $f(U)$ is open (and connected).

  - for all $z \in U$ there is a $u \in U$ such that
    $$|f(u)| > |f(z)|$$
    that means $|f|$ does not have a maximum at any point of U
    - minimum can only be 0 because otherwise the modulus of the holomorphic
      $$\frac{1}{f(z)}$$
      attains a maximum.
  - In other words,
    $$\underbrace{U \xrightarrow{f} \mathbb{C}}_{\text{open}} \underbrace{\xrightarrow{|\,|}}_{\text{open}} [0, \infty)$$
    implies $|f(U)|$ is *open* in $[0, \infty)$.

Therefore,

> **(Maximum of holomorphic functions on is never attained on the interior)** Let $f \in \mathscr{O}(U)$ be a **non-constant** holomorphic function on a open set $U \subseteq \mathbb{C}$. Then $|f|$ has no local maximum on U.
> For $f \in \mathscr{O} \cap \mathcal{C}(\overline{U})$ where $U \subseteq \mathbb{C}$ is a bounded open subset, $\|f\|_\infty$ is attained on ∂U.

via harmonic functions

> *Steady-state thermal* interpretation of Maximum of holomorphic functions on is never attained on the interior
>
> As
> $$z \mapsto \log|f(z)|$$
> is a harmonic function, steady-state temperature, there cannot be any maximum on the interior of a domain.

via Cauchy's integral formula

- Let $z_0 \in U$ is such that
  $$|f(z_0)| \geq |f(z)| \qquad z \in U$$

- $$\begin{aligned} f(z_0) &= \frac{1}{2\pi i}\int_{\partial B(r,z_0)} \frac{f(z)}{z - z_0}\mathrm{d}z \\ \implies |f(z)| \leq |f(z_0)| &= \frac{1}{2\pi}\left|\int_{t\in[0,2\pi]} f(z + re^{it})\mathrm{d}t\right| \\ &\leq \frac{1}{2\pi}\int_{t\in[0,2\pi]} \left|f(z + re^{it})\right|\mathrm{d}t \end{aligned}$$
  - As $r \to 0$ the average converges to $|f(z)|$ by

    > Let $g \in \mathcal{C} \cap L^1(\mathbb{R}^n)$. Then for every $x \in \mathbb{R}^n, \delta > 0$ there exists $r(\delta, x) > 0$ such that
    > $$\forall y \in B_{r(\delta,x)}(x), |g(y) - g(x)| < \delta$$
    > - For all $x \in \mathbb{R}^n, \epsilon > 0$, if $r < r(\delta, x)$ we have $y \in B_r(x) \subseteq B_{r(\delta,x)}$ then
    > $$\begin{aligned} |(A_r g - g)(x)| &= \frac{1}{m(B_r(x))}\left|\int_{B_r(x)} (g - g(x))\right| \\ &\leq \frac{1}{m(B_r(x))}\int_{B_r(x)} \underbrace{|g - g(x)|}_{<\delta} \\ &< \delta \end{aligned}$$
    > - Therefore,
    > $$\lim_{r\to 0}(A_r f)(x) = f(x)$$
    > for all $x \in \mathbb{R}^n$.
  - *or*, we have
    $$0 \leq \int_{t\in[0,2\pi]} \left(|f(z_0)| - \left|f(z_0 + re^{it})\right|\right) \leq 0$$
    which implies...?
- By

  > **(Identity theorem for holomorphic functions on subsets of $\mathbb{C}$)** Let
  > $$f: \Omega(\text{open}) \subseteq \mathbb{C} \to \mathbb{C}$$
  > be holomorphic. And L be the set of limit points of $f^{-1}(0)$. Then L **is both open and closed in** Ω.

> Therefore if Ω is connected, either $L=\Omega \iff f\equiv 0$ on Ω or $f^{-1}(0)$ has no limit point in Ω, that is if $f^{-1}(0)$ has a limit point in Ω then $f\equiv 0$.

$$f \equiv f(z_0) \qquad \text{on } U$$

on difference of holomorphic functions

For two holomorphic functions f,g on a connected open set U we get $f-g$ on U which can either be constant or have no maximum modulus on U,... this probably tells us nothing.

on derivatives

Let $f\in\mathscr{O}(U)$. Then some higher derivative $f^{(n)}$ attains a maximum $\iff$ f is a polynomial of degree $\le n$.

Current note has 0 direct children and 0 total descendants.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $\Gamma\backslash\mathbb{R}^n, \Gamma\backslash\mathbb{C}^n$
    - Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - mapping
        - Maximum and minimum of holomorphic functions

And it has 13 siblings.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $\Gamma\backslash\mathbb{R}^n, \Gamma\backslash\mathbb{C}^n$
    - Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - mapping
        - Bounding derivative of holomorphic mappings
        - Bounding holomorphic functions linearly
        - Branch points and ramification index of holomorphic functions
        - Disk in holomorphic image of disk
        - Size of holomorphic image of disks
        - Maximum and minimum of holomorphic functions
        - Fibers of holomorphic mappings
        - Non-constant holomorphic functions locally look like w^k and are open mappings
        - Biholomorphic mapping onto disk
        - Conformal radius of pointed simply connected open subsets of $\mathbb{C}$
        - Holomorphic functions on punchered disk
        - Holomorphic function whose singularities are not isolated
        - A holomorphic map branched over $\mathbb{Q}$