Info

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Non-constant holomorphic maps between Riemann surfaces locally looks like w^k and are open maps

(Open mapping theorem for Riemann surfaces) Let $f : X \to Y$ be a non-constant holomorphic map between Riemann surfaces. Then f is an open map.

non-constant holomorphic function locally looks like w^k on a change of coordinates

> **Proposition:** Let
>
> $$f : X \to Y$$
>
> be a non-constant holomorphic map between Riemann surfaces and $f(a) = b$ for some $a \in X, b \in Y$. Then there is an integer $k \geq 1$ and charts (U, φ) and (V, ψ) on X, Y respectively such that
>
> - $a \in U, \varphi(a) = 0$ and $b \in V, \psi(b) = 0$
> - $f(U) \subseteq V$
> - $$(\psi \circ f \circ \varphi^{-1})(z) = z^k \text{ on } V$$

From

non-constant holomorphic function locally looks like w^k on a change of coordinates

Let we have a holomorphic function f on some open subset of $\mathbb{C}$ containing z_0.

> **Proposition:** If $f(z_0) \neq 0$ and *any* positive integer k, we compose f with the function $z^{1/k}$ to obtain a function h such that
>
> $$h^k = f \text{ on } V$$
>
> for a smaller open V containing z_0.

> **Proposition:** If $f : U \to \mathbb{C}$ is a *non-constant* holomorphic function on a open subset $U \subseteq \mathbb{C}$,
>
> - then around any $z_0 \in U$ there is a holomorphic $g(z)$
>
> $$\begin{aligned} f(z) = f(0) + (z - z_0)^k g(z) &\text{ around } z_0 \\ g(z_0) &\neq 0 \end{aligned}$$
>
> - Moreover there is a smaller neighborhood around z_0 and a holomorphic $h(z)$ such that
>
> $$\begin{aligned} f(z) = f(0) + ((z - z_0)h(z))^k &\text{ around } z_0 \\ h(z_0) &\neq 0 \end{aligned}$$
>
> with further smaller neighborhood around z_0 so that $(z - z_0)h(z)$ is a biholomorphism to its (open) image.

- Now, we write

$$f(z) = \sum_{n \geq 0} a_n (z - z_0)^n \text{ on } z \in B_R(z_0)$$

- If f is **not constant**, there is some smallest $k \geq 1$ (**multiplicity of the zero**) such that $a_k \neq 0$, meaning

$$\begin{aligned} f(z) = a_0 + (z - z_0)^k g(z) &\text{ on } z \in B_R(z_0) \\ g(z) &= \sum_{n \geq 0} a_{n+k} (z - z_0)^n \\ g(z_0) &\neq 0 \end{aligned}$$

- Then by

  > **Proposition:** If $f(z_0) \neq 0$ and *any* positive integer k, we compose f with the function $z^{1/k}$ to obtain a function h such that
  >
  > $$h^k = f \text{ on } V$$
  >
  > for a smaller open V containing z_0.

  we find a smaller open V containing z_0 and a holomorphic h such that

$$\begin{aligned} f(z) = a_0 + (z - z_0)^k (h(z))^k &\text{ on } z \in V \\ h(z_0) &\neq 0 \end{aligned}$$

  - Thus we have

  $$f(z) = a_0 + (\underbrace{(z - z_0)h(z)}_{h_1(z)})^k \text{ on } z \in V$$

  with $h(z_0) \neq 0$, so

  $$\begin{aligned} h_1(z) &= (z - z_0)h(z) \\ h_1'(z) &= h(z) + (z - z_0)h'(z) \\ h_1'(z_0) &= h(z_0) \neq 0 \end{aligned}$$

  - As h_1 has a non-vanishing derivative at z_0, so there must be *another* smaller open W containing z_0 where $h_1(W)$ is open and

$$h_1 : W \to h_1(W)$$

is a homeomorphism.

- We can choose W so that $h_1(W)$ is a disk centered at $h_1(z_0) = 0$.

Proposition:

$$f : W \to f(W) = a_0 + D$$
$$f(z) = a_0 + (h_1(z))^k$$

is an **open** map.

- Given open set $A \subseteq W$ its image $h_1(A)$ under the homeomorphism h_1 is open set in a disk containing 0.
- Image of $h_1(A)$ under $z \mapsto z^k$ is open and translation $w \mapsto a_0 + w$ takes open sets to open sets.
- Thus $f(A)$ is open.

we conclude from any charts around a, b we may construct the charts desired.

non-constant holomorphic maps between Riemann surfaces are open

(Open mapping theorem for Riemann surfaces) Let $f : X \to Y$ be a non-constant holomorphic map between Riemann surfaces. Then f is an open map.

- If $f : X \to Y$ is a holomorphic map between Riemann surfaces X, Y and X is **compact** then by

  (Open mapping theorem for Riemann surfaces) Let $f : X \to Y$ be a non-constant holomorphic map between Riemann surfaces. Then f is an open map.

  we know

  $$f(X)$$

  is compact $\implies$ closed and open and because Y is connected. Therefore,

  $$f(X) = Y$$

  that is f is **surjective** and $f(X) = Y$ is also **compact**.

Current note has 0 direct children and 0 total descendants.

And it has 3 siblings.