Info

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D_n

Definition. For $n \geq 3$, D_n is the set of symmetries (isometries of $\mathbb{R}^2$ that keep the subset invariant) of a regular n-gon, with the operation of composition of function.

It is called *dihedral* because it is "generated by two plane mirrors".

- Center of the dihedral group

description of D_n

$$D_n = \{i, r, r^2, \ldots, r^{n-1}, s, rs, \ldots, r^{n-1}s\}$$

- for $0 \leq i < n$ (includes i)
  - all of order n
  - has no fixed point on the n-gon
- n reflections

$$r^i s$$

for $0 \leq i < n$
  - all of order 2
  - has fixed points?

presentations

$$D_n = \langle r, s | r^n = i = s^2, srs^{-1} = r^{-1} \rangle$$

$$D_n = \langle a, b | a^2 = b^2 = (ab)^n = i \rangle$$

$D_n \to GL_{\mathbb{R}}(2)$

$D_n \cong_{\mathsf{Grp}} D_{\frac{n}{2}} \times \mathbb{Z}_2$

representations

$$D_{2n} = \langle r, s \mid r^n = s = i,\ srs = r^{n-1} \rangle$$

Let

$$D_{2n} \longrightarrow GL_{\mathbb{C}}(V)$$

be irreducible. Then the restriction

(1) $\mathbb{Z}_n \cong \langle r \rangle \longrightarrow GL_{\mathbb{C}}(V)$

must break into 1-dim inv. subspaces

$$V \cong \underset{r \circlearrowright \lambda_1}{\mathbb{C}v_1} \oplus \underset{r \circlearrowright \lambda_2}{\mathbb{C}v_2} \oplus \cdots \oplus \underset{r \circlearrowright \lambda_m}{\mathbb{C}v_m}$$

then

$$\cdots \oplus \underset{r \circlearrowright \lambda_i}{\mathbb{C}v_i} \oplus \cdots \Rightarrow \cdots \oplus \underset{r^k \circlearrowright \lambda_i^k}{\mathbb{C}v_i} \oplus \cdots$$

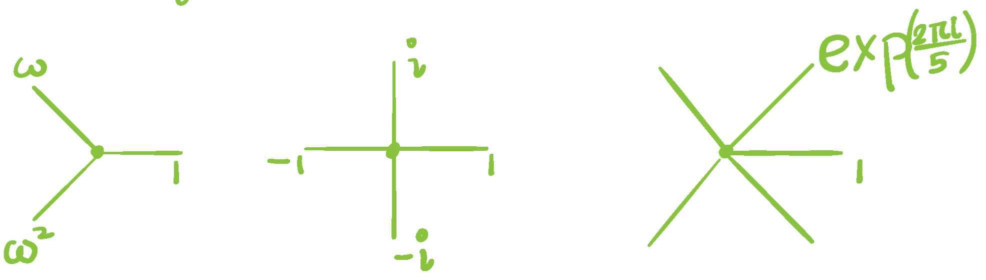
$$r^n = I \Rightarrow \lambda_i^n = 1$$

(2) Now, we must ask about $s : V \to V$

$$srs = r^{n-1}$$
$$\Rightarrow rs = sr^{n-1}$$
$$\Rightarrow rs(v_i) = sr^{n-1}(v_i)$$
$$\Rightarrow r\left(s(v_i)\right) = \lambda_i^{n-1}\left(s(v_i)\right)$$

So,

$$\cdots \oplus \underset{r \circlearrowright \lambda_i}{\mathbb{C}v_i} \oplus \cdots \underset{s}{\overset{s}{\rightleftarrows}} \oplus \underset{r \circlearrowright \lambda_i^{n-1}}{\mathbb{C}s(v_i)} \oplus \cdots$$



n=3 n=4 n=5

if $\lambda_i = 1, -1$

and if $s(v_i) \in \mathbb{C}v_i$

$\mathbb{C}v_i$: $s \circlearrowright$ 1 or -1, $r \circlearrowright$ 1 or -1 (n even)

- if n is odd

$s \backslash r \mapsto$	I
I	1
$-I$	2

2 reps.

- if n is even

$s \backslash r \mapsto$	I	$-I$
I	1	3
$-I$	2	4

4 reps.

- if $s(v_i) \notin \mathbb{C}(v_i)$

$$\underset{r \circlearrowright 1}{\mathbb{C}v_i} \oplus \underset{r \circlearrowright 1}{\mathbb{C}s(v_i)} \quad (s \text{ swapping})$$

$$\cong \underset{r \circlearrowright 1,\ s \circlearrowright -1}{\mathbb{C}\{v_i - s(v_i)\}} \oplus \underset{r \circlearrowright 1,\ s \circlearrowright 1}{\mathbb{C}\{v_i + s(v_i)\}}$$

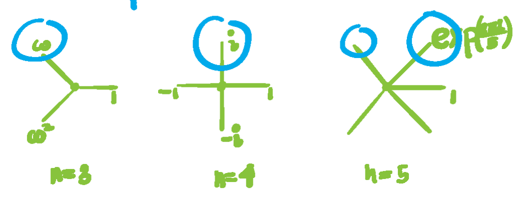
reducible

if $\lambda_i \neq 1, -1$

$$\underset{r \circlearrowright \lambda_i}{\mathbb{C}v_i} \underset{s}{\overset{s}{\rightleftarrows}} \oplus \underset{r \circlearrowright \lambda_i^{n-1}}{\mathbb{C}s(v_i)}$$

is a 2-dim inv subspace.
$\Rightarrow$ it is V.

Choose λ_i from upper half plane.



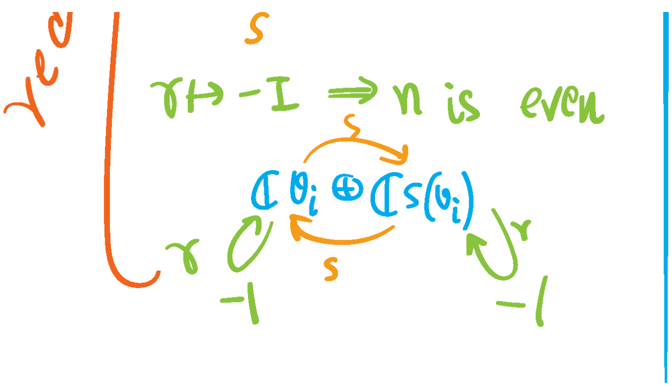
$\frac{n-1}{2}$ if n is odd

$\frac{n-2}{2}$ if n is even.

$$\underset{r \circlearrowright \lambda}{\mathbb{C}v} \underset{s}{\overset{s}{\rightleftarrows}} \oplus \underset{r \circlearrowright \lambda^{n-1}}{\mathbb{C}s(v)}$$

$$r \longmapsto \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{n-1} \end{bmatrix}$$

$$s \longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



[1]

Current note has 2 direct children and 2 total descendants.

- stretchy
  - dihedral
    - D_n
      - Center of the dihedral group
      - From the symmetries of a n-gon

And it has 1 siblings.

- stretchy
  - dihedral
    - D_n

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1. 2402-irrreps-1.pdf ↩