ⓘ Info

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Geodesic flow of Fuchsian groups and hyperbolic surfaces

$$\begin{tikzcd}
& PSL(2,\mathbb{R}) \arrow[rr] \arrow[d] \arrow[ddl] & & PSL(2,\mathbb{R})/SO(n,\mathbb{R}) \arrow[d] \arrow[ddl] \\
& T^u\mathbb{R}H^n \arrow[rr] \arrow[ddl] & & \mathbb{R}H^n \arrow[ddl] \\
{}_{\Gamma}\backslash^{PSL(2,\mathbb{R})} \arrow[rr] \arrow[d] & & {}_{\Gamma}\backslash^{PSL(2,\mathbb{R})}/SO(n,\mathbb{R}) \arrow[d] & \\
X := T^u M \arrow[rr] & & M = {}_{\Gamma}\backslash^{\mathbb{R}H^n} &
\end{tikzcd}$$

> Let $\Gamma \leq PSL(2)(\mathbb{R})$ be a lattice. Then any $g(t) := R_{a_t}$ for some $t \neq 0$ is an ergodic transformation on
>
> $$\left(X := {}_{\Gamma}\backslash^{PSL(2)(\mathbb{R})}, \nu_X\right)$$

$$a_t := \begin{bmatrix} \exp\left(-\frac{t}{2}\right) & \\ & \exp\left(\frac{t}{2}\right) \end{bmatrix}$$
$$u^- := \begin{bmatrix} 1 & s \\ & 1 \end{bmatrix}$$

We fix a $t \neq 0$ and normalize the Haar measure ν_X to ensure $\nu_X(X) = 1$.

- Let
  $$f : X \to \mathbb{R}$$
  be a measurable $R_{a_t} =: R$-invariant function.
- For any $\epsilon > 0$, by
  > **(Lusin)** Let f be measurable function on $E \subseteq \mathbb{R}^d$ with $m(E) < \infty$. Then for every $\epsilon > 0$ there exists a closed set $F_\epsilon \subset E$ such that $m(E - F_\epsilon) \leq \epsilon$ so that
  >
  > $$f\Big|_{F_\epsilon}$$
  >
  > is continuous.

  choose a compact set $K \subseteq X$ with measure $\nu_X(K) > 1 - \epsilon$ such that
  $$f|_K$$
  is continuous.

steps	roughly
Proposition: Fix $\epsilon > 0$. Let $K \subseteq X$ be a compact set of measure $\nu_X(K) > 1 - \epsilon$ $$B := \left\{ x \middle\| \lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n-1} \chi_K(R^l x) > \frac{1}{2} \right\}$$ has measure $\nu_X(B) \geq 1 - 2\epsilon$	more than $\frac{1}{2}$ of future of $x \in B$ belongs to K
• Then because $$f(-) = f(R^l(-))$$ for x, y and for all $l \geq 1$ we have $$d_X(R^l(x), R^l(y)) = d_X(xa_t^{-l}, xu^-(-s)a_t^l) \leq d_{PSL(2)(\mathbb{R})}(I_2, a_t^l u^-(-s)a_t^{-l}) \xrightarrow{l\to\infty} 0$$	simultaneous times $l \geq 0$ with $R^l x, R^l y \in K$
	$$R^n(x) = R^n(x)a_t^n u^- a_t^{-n},\ R^n(y)$$ are close together for $n \gg 1$ so $$f(x) = f(R^n(x)) \approx f(R^n(y)) = f(y)$$

> **Proposition:** Fix $\epsilon > 0$. Let $K \subseteq X$ be a compact set of measure $\nu_X(K) > 1 - \epsilon$
>
> $$B := \left\{ x \middle| \lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n-1} \chi_K(R^l x) > \frac{1}{2} \right\}$$
>
> has measure $\nu_X(B) \geq 1 - 2\epsilon$

- By
  > **(Pointwise ergodic theorem for measure preserving transformation)** Let $(X, \mathfrak{M}_X, \mu)$ be a probability space, $T : X \to X$ be μ-preserving and $f \in L^1(X, \mu)$. Then
  >
  > $$\langle f \rangle_T = \lim_{n\to\infty} A_T^{(n)}(f) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k$$

converges pointwise almost everywhere on X to a T-invariant function $\langle f \rangle_T \in L^1(X,\mu)^T$ and

$$\int_X \langle f \rangle_T \mathrm{d}\mu = \int_X f \mathrm{d}\mu$$

$$\begin{aligned}\chi_K^* : X &\to [0,1] \\ x &\mapsto \lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n-1} \chi_K(R^l x)\end{aligned}$$

exists almost everywhere, then

$$B = \left\{ \infty > \chi_K^* > \frac{1}{2} \right\}$$

and

$$\int_X \chi_K^* \mathrm{d}\nu_X = \nu_X(K) \geq 1 - \epsilon$$

- Thus,

$$\nu_X(B) \geq 1 - 2\epsilon$$

- So,

$$\begin{aligned}1 - \epsilon \leq \int_B \chi_K^* \mathrm{d}\nu_X + \int_{X\setminus B} \chi_K^* \mathrm{d}\nu_X &\leq \nu_X(B) + \frac{1}{2}\nu_X(X\setminus B) \\ &= \nu_X(B) + \frac{1}{2}(1 - \nu_X(B)) \\ &= \frac{1}{2}\nu_X(B) + \frac{1}{2}\end{aligned}$$

- Thus

$$\nu_X(B) \geq 1 - 2\epsilon$$

- Let $x, y := xu^-(s) \in B$ for some $s \in \mathbb{R}$.
  - Then because

$$f(-) = f(R^l(-))$$

  for x, y and for all $l \geq 1$ we have

$$\begin{aligned}d_X(R^l(x), R^l(y)) &= d_X(xa_t^{-l}, xu^-(-s)a_t^l) \\ &\leq d_{PSL(2)(\mathbb{R})}(I_2, a_t^l u^-(-s) a_t^{-l}) \xrightarrow{l\to\infty} 0\end{aligned}$$

  By

  > **Proposition:** Fix $\epsilon > 0$. Let $K \subseteq X$ be a compact set of measure $\nu_X(K) > 1 - \epsilon$
  >
  > $$B := \left\{ x \middle| \lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n-1} \chi_K(R^l x) > \frac{1}{2} \right\}$$
  >
  > has measure $\nu_X(B) \geq 1 - 2\epsilon$

  we have a common sequence $l_n \xrightarrow{n\to\infty} \infty$ such that

$$R^{l_n}(x), R^{l_n}(y) \in K$$

  - **Proposition:**

$$x, xu^-(s) \in B \implies f(x) = f(xu^-(s))$$

> **Proposition:**
>
> $$f(x) = f(xg)$$
>
> for all $g \in PSL(2)(\mathbb{R})$ and for all
>
> $$x \in X_g := \dots$$
>
> where X_g as full measure.

- Assume

$$f : X \to \mathbb{R}$$

  is **not** constant ν_X-ae.
  - Then there exist disjoint intervals $I_1, I_2 \subseteq \mathbb{R}$ such that

$$C_j := \{h \in PSL(2)(\mathbb{R}) | f(\Gamma h) \in I_j\}$$

  for $j = 1, 2$ which are neither $\nu_{PSL(2)(\mathbb{R})}$-measure zero or of full $\nu_{PSL(2)(\mathbb{R})}$-measure.

  There is a $g \in G$ such that

$$\nu(C_1 \cap C_2 g) > 0$$

  - But

$$D_g := \pi_\Gamma^{-1} X_g$$

  has full measure, so there is some

$$h \in C_1 \cap C_2 g \cap D_g$$

  - But this contradicts f being constant on X_g.
  - Thus f is constant ν_X-ae.

Current note has 0 direct children and 0 total descendants.

- stem
    - Objects stemming from $PSL(2)(\mathbb{R})$
        - Fuchsian groups $\Gamma <_\mathrm{d} PSL(2)(\mathbb{R})$ $\leftrightarrow$ oriented hyperbolic 2-manifold/orbifolds $\to$ Riemann surfaces with branching signature
            - Geodesic flow of Fuchsian groups and hyperbolic surfaces

And it has 17 siblings.

- stem
    - Objects stemming from $PSL(2)(\mathbb{R})$
        - Fuchsian groups $\Gamma <_\mathrm{d} PSL(2)(\mathbb{R})$ $\leftrightarrow$ oriented hyperbolic 2-manifold/orbifolds $\to$ Riemann surfaces with branching signature
            - Artihmetic Fuchsian groups and Artihmetic hyperbolic 2-orbifolds
            - Cocompact Fuchsian groups
            - Central simple algebra
            - Elementary Fuchsian groups
            - Finitely generated Fuchsian groups $\leftrightarrow$ geometrically finite Fuchsian groups
            - Fuchsian groups of finite coarea
            - Geodesic flow of Fuchsian groups and hyperbolic surfaces
            - Decomposition of hyperbolic surfaces
            - Infinitely generated Fuchsian groups
            - Limit set of Fuchsian groups
            - Marked hyperbolic surfaces
            - subst.R PSL 2 discrete subg.over Q-bar
            - Fuchsian Schottky groups
            - Spectrum of hyperbolic surfaces
            - Torsion-free Fuchsian groups
            - Cocompact torsion-free Fuchsian groups
            - Fuchsian triangle groups