Info

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Uniform boundedness of a famly of linear maps

(Banach-Steinhaus uniform boundedness principle) Let X be a Banach space and Y be a normed linear space. Let $\mathcal{F} \subseteq \mathcal{B}(X, Y)$ be a family of bounded linear maps. Then either $\mathcal{F}$ is *uniformly bounded*

$$\sup_{T\in\mathcal{F}} \|T\|_{X\to Y} < \infty$$

or there exists a dense subset $S \subseteq X$ such that for each $x \in S$

$$\sup_{T\in\mathcal{F}} \|T(x)\|_Y = \infty$$

(weaker uniform bounded principle) Let X be a Banach space and Y be a normed linear space. Let $\mathcal{F} \subseteq \mathcal{B}(X, Y)$ be a family of bounded linear maps. Then the family $\mathcal{F}$ is pointwise bounded on X

$$\forall x \in X, \sup_{T\in\mathcal{F}} \|T(x)\|_Y < \infty$$

$\iff$ $\mathcal{F}$ is pointwise bounded on the unit ball of X

$$\forall x \in X : \|x\|_X < \infty, \sup_{T\in\mathcal{F}} \|T(x)\|_Y < \infty$$

$\iff$ $\mathcal{F}$ is uniformly bounded

$$\sup_{T\in\mathcal{F}} \|T\|_{X\to Y} < \infty$$

(stronger uniform bounded principle) Moreover, if $\mathcal{F}$ is unbounded at a point $x_0 \in X$, then it is not bounded on a dense open subset of X.

as Banach spaces are Baire

Consider

Definition.
$$S_N := \{x \in X | \exists T \in \mathcal{F} : \|T(x)\|_Y > N\}$$

- Either one of these open sets are not dense.
  - Let S_k is not dense in X for some $k \geq 1$. Then there exists $x_0 \in X$ and $r > 0$ such that

    $$\overline{B(x_0, r)} \cap S_k = \emptyset$$
    $$\iff \forall T \in \mathcal{F}, x \in \overline{B(x_0, r)}, \|T(x)\| \leq k$$

  - For $\|z\|_X \leq r$ we have

    $$\begin{aligned}\|T(z)\|_Y &= \|T(x_0 + z) - T(x_0)\|_Y \\ &\leq \|T(x_0 + z)\|_Y + \|T(x_0)\| \\ &\leq 2k\end{aligned}$$

  - Then

    $$\|T\|_{X\to Y} = \sup_{x\in X, x\neq 0} \frac{\|T(x)\|_Y}{\|x\|_X} = \sup_{\|x\|_X\leq r} \frac{\|T(x)\|_Y}{r} \leq \frac{2k}{r}$$

  - Hence, the family of operators $\mathcal{F}$ is uniformly bounded.
- Otherwise each of these open sets are dense.
  - By

    (complete metric spaces are Baire) Let $\{S_k\}_{k\geq 1}$ be a sequence of open, dense subsets of a nonempty complete metric space X. Then the intersection

    $$\bigcap_{k\geq 1} S_k$$

    is a nonempty, dense subset of X.

    $$S := \bigcap_{k\geq 1} S_k$$

    is also dense.
  - This implies for each $x \in S$ and $n \geq 1$ there exists $\Lambda \in \mathcal{F}$ such that $\|T(x)\|_Y > n$.
  - Therefore, for each $x \in S$

    $$\sup_{T\in\mathcal{F}} \|T(x)\|_Y = \infty$$

pointwise bounded $\implies$ uniformly bounded without using Baire

Suppose $\sup_{T\in\mathcal{F}} \|T\| = \infty$.

- Choose $\{T_n\}_{n\geq 1} \subset \mathcal{F}$ such that

  $$\|T_n\| \geq 4^n$$

- **Base case:** Let $x_0 := 0$

- **Induction hypothesis:** For $n > 0$ we have a $x_n \in X$ such that

$$\begin{aligned} \|x_n - x_{n-1}\| &\leq 3^{-n} \\ \|T_n(x_n)\| &\geq \frac{2}{3} 3^{-n} \|T_n\| \end{aligned}$$

- **Inductive step:** Pick any x_{n+1} such that

$$\|x_{n+1} - x_n\| < \frac{2}{3} \frac{1}{3^{n+1}} \leq \frac{1}{3^{n+1}}$$

and then by

> Let $T : X \to Y$ be a bounded linear map between normed linear spaces X, Y. Then for any $x \in X, r > 0$ we have
>
> $$\sup_{z \in B(x,r)} \|T(z)\| \geq \|T\| r$$

we have

$$\|T_n(x_{n+1})\| \geq \frac{2}{3} \frac{1}{3^{n+1}} \|T_n\|$$

- As

$$\|x_{n+1} - x_n\| \leq \frac{1}{3^n}$$

sequence $\{x_n\}$ is Cauchy and hence converges to some $x_\infty \in X$.

- $$\|x_\infty - x_n\| \leq \sum_{k \geq n} \frac{1}{3^k} = \frac{1}{2} \frac{1}{3^{n-1}}$$

- Hence, again by

> Let $T : X \to Y$ be a bounded linear map between normed linear spaces X, Y. Then for any $x \in X, r > 0$ we have
>
> $$\sup_{z \in B(x,r)} \|T(z)\| \geq \|T\| r$$

$$\|T_n(x_\infty)\| \geq \frac{1}{6} 3^{-n} \|T_n\| \geq \frac{1}{6} \left(\frac{4}{3}\right)^n \xrightarrow{n \to \infty} \infty$$

[1]

[2]

Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - Topological spaces
    - Topological vector spaces over $\mathbb{R}$ or $\mathbb{C}$
      - map
        - Uniform boundedness of a famly of linear maps

And it has 3 siblings.

- structures based on sets
  - Topological spaces
    - Topological vector spaces over $\mathbb{R}$ or $\mathbb{C}$
      - map
        - Image of balls under linear map between topological vector spaces
        - Limit of a sequence of linear maps
        - Uniform boundedness of a famly of linear maps

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1. https://arxiv.org/pdf/1005.1585 ↩
2. https://www.imsc.res.in/~kesh/trinity-sequel.pdf ↩