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Discrete subgroups of $\mathbb{R}^n$

Let $\Gamma \leq \mathbb{R}^n$ be a subgroup and $V := \mathbb{R}\langle \Gamma \rangle$ be its $\mathbb{R}$ -span. Then Γ is **discrete** $\iff$ there is a $\mathbb{R}$ -linear isomorphism

$$T : \mathbb{R}^{\dim V} \cong_{\text{Vec}_{\mathbb{R}}} V$$

such that $T : \mathbb{Z}^{\dim V} \cong_{\text{Ab}} \Gamma$.

Let $\Gamma \leq \mathbb{R}^n$ be a discrete subgroup and $V := \mathbb{R}\langle \Gamma \rangle$ be its $\mathbb{R}$ -span with $\dim V =: k$. Consider a $\mathbb{R}$ -basis $\{\gamma_1, \dots, \gamma_k\}$ of V .

- Let $\Gamma_0 := \mathbb{Z}\langle \gamma_1, \dots, \gamma_k \rangle \leq \Gamma$.

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$$\begin{array}{ccc} V & \rightarrow & \underbrace{V/\Gamma_0}_{\text{compact}} \\ & \searrow & \downarrow \\ & & V/\Gamma \end{array}$$

is a commutative diagram of covering maps with V/Γ_0 being homeomorphic to $\mathbb{R}^k/\mathbb{Z}^k$ which is compact.

- Thus

$$\begin{array}{ccc} \underbrace{\Gamma/\Gamma_0}_{\text{finite}} & \hookrightarrow & \underbrace{V/\Gamma_0}_{\text{compact}} \\ & & \downarrow \text{finite degree} \\ & & V/\Gamma \end{array}$$

we conclude Γ/Γ_0 is finite.

- Therefore, Γ is finitely generated, torsion free Abelian group with a finite index subgroup Γ_0 of rank k . By

Let G be a free Abelian group with rank n and $H \leq G$ be a subgroup. Then H is a free Abelian group with rank $n \iff H \leq G$ is of finite index.

we conclude Γ also has rank $k = \dim V$.

- Thus

$$\Gamma = \mathbb{Z}\langle \alpha_1, \dots, \alpha_k \rangle$$

such that

$$V \supseteq \mathbb{R}\langle \alpha_1, \dots, \alpha_k \rangle \supseteq \mathbb{R}\langle \gamma_1, \dots, \gamma_n \rangle = V$$

- We construct a linear isomorphism

$$\begin{array}{l} \mathbb{R}^{\dim V} \cong V \\ e_i \mapsto \alpha_i \end{array}$$

which takes $\mathbb{Z}^{\dim V}$ to Γ .

Current note has 1 direct children and 1 total descendants.

- [stem](#)
 - [subobjects of and functions on \$\mathbb{R}^n, S^n, \mathbb{C}^n\$ and its quotients \$\Gamma \backslash \mathbb{R}^n, \Gamma \backslash \mathbb{C}^n\$](#)
 - [Discrete subgroups of \$\mathbb{R}^n\$](#)
 - [Power series sum of a discrete subgroup of \$\mathbb{R}^2\$](#)

And it has 39 siblings.

- [stem](#)
 - [subobjects of and functions on \$\mathbb{R}^n, S^n, \mathbb{C}^n\$ and its quotients \$\Gamma \backslash \mathbb{R}^n, \Gamma \backslash \mathbb{C}^n\$](#)
 - [Holomorphic functions on spaces over \$\mathbb{C}\$ of dimension 1](#)
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 - [Holomorphic subsets of \$\mathbb{C}^n\$](#)
 - [Regular hypersurfaces in \$\mathbb{R}^{2n}\$](#)

- Orientable hypersurfaces in $\mathbb{R}^n$
- Laplace operator on $\mathbb{R}^n$
- $l^\infty(\mathbb{R})$
- Lebesgue measurable subsets of and functions on $\mathbb{R}^n, T^n, S^n$
- Lebesgue measure of boundary of open sets in $\mathbb{R}^n$
- Local $\mathbb{R}$ -algebras
- $l^p(\mathbb{R})$
- Metric density of subsets of $\mathbb{R}^n$
- Monotone functions on $\mathbb{R}$
- Cauchy integral of periodic functions
- Polygons with integer vertices
- Open smooth maps $U \subseteq \mathbb{R}^2 \rightarrow \mathbb{C}$
- Discrete subgroups of $\mathbb{R}^n$
- Discrete cocompact subgroups of $\mathbb{R}^n$, flat tori
- Ramified germs of smooth and holomorphic functions
- Open subsets of $\mathbb{R}^n$
- Open subsets of $\mathbb{R}^n$ equipped with the flat metric
- Closed subsets of $\mathbb{R}^n$ equipped with smooth functions
- Quasi-analytic smooth functions on $\mathbb{R}$
- Star-shaped subsets of $\mathbb{R}^n$
- ODEs in $\mathbb{R}^n$ $\leftrightarrow$ Vector fields in $\mathbb{R}^n$