Info

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is written (completely with human hands) by Rupadarshi Ray,
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Differential forms on complex 1-manifolds

These are smooth complex differential forms on a Riemann surface X

$$\begin{array}{ccccc} \Omega^0(X,\mathbb{C}) & \to & \Omega^{1,0}(X,\mathbb{C}) \oplus \Omega^{0,1}(X,\mathbb{C}) & \to & \Omega^2(X,\mathbb{C}) \\ \phi & \overset{\mathrm{d}}{\mapsto} & \mathrm{d}\phi = \frac{\partial\phi}{\partial z}\mathrm{d}z + \frac{\partial\phi}{\partial\bar{z}}\mathrm{d}\bar{z} & & \\ & & f\mathrm{d}z + g\mathrm{d}\bar{z} & \overset{\mathrm{d}}{\mapsto} & \left(\frac{\partial g}{\partial z} - \frac{\partial f}{\partial \bar{z}}\right)\mathrm{d}z\wedge\mathrm{d}\bar{z} \\ & & & & h\mathrm{d}z\wedge\mathrm{d}\bar{z} \end{array}$$

These are $\mathcal{C}^\infty(M,\mathbb{C})$-differential forms on a Riemann surface and the exterior derivative.

The Hodge diamond of Riemann surfaces

$$\begin{array}{ccc} & h^{0,0} & \\ h^{1,0} & & h^{0,1} \\ & h^{1,1} & \end{array}$$

Hodge diamonds

Proposition:

- $$H^{0,0} = \frac{\Omega^{0,0}\cap\ker\bar{\partial}}{0} = \Omega^0_{\mathscr{O}} = \mathscr{O}$$
- $$H^{0,1} = \frac{\Omega^{0,1}}{\bar{\partial}\Omega^{0,0}} = 0$$
- $$H^{1,0} = \frac{\Omega^{1,0}\cap\ker\bar{\partial}}{0} = \Omega^1_{\mathscr{O}}$$
- $$H^{1,1} = \frac{\Omega^{1,1}}{\bar{\partial}\Omega^{1,0}} = 0$$

Proposition:

- For a connected **compact** Riemann surface, the Hodge diamond looks like
$$\begin{array}{ccc} & h^{0,0} & \\ h^{1,0} & & h^{0,1} \\ & h^{1,1} & \end{array} = \begin{array}{ccc} & 1? & \\ g & & g \\ & 0? & \end{array}$$
- For a connected **non-compact** Riemann surface, the Hodge diamond looks like
$$\begin{array}{ccc} & h^{0,0} & \\ h^{1,0} & & h^{0,1} \\ & h^{1,1} & \end{array} = \begin{array}{ccc} & \infty? & \\ \infty? & & 0 \\ & 0 & \end{array}$$

Holomorphic forms

Proposition:

- For a connected **non-compact Riemann surface** M
$$H^1(\Omega^\bullet_{\mathscr{O}}(M)) = \frac{\Omega^1_{\mathscr{O}}}{\mathrm{d}\Omega^0_{\mathscr{O}}} \cong H^1(M,\mathbb{C})$$
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Current note has 0 direct children and 0 total descendants.

- stem
  - Objects stemming from $\mathbb{R}$-spaces of dimension 2
    - Riemann surfaces
      - T C
        - Differential forms on complex 1-manifolds

And it has 2 siblings.

- stem
  - Objects stemming from $\mathbb{R}$-spaces of dimension 2
    - Riemann surfaces
      - T C
        - $\boldsymbol{Sym}^2(T^{(1,0)}X) \twoheadrightarrow X$
        - Differential forms on complex 1-manifolds