Info

This note, that was found here, as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
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Isometric embedding of a subset of $\mathbb{R}^n$ into $\mathbb{R}^n$

Let

$$\begin{aligned} f : S \subseteq \mathbb{R}^n \to \mathbb{R}^n \\ f(0) = 0 \\ \|f(x) - f(y)\| = \|x - y\| \end{aligned}$$

be a map.

Then for $y = 0$ we have

$$\|f(x)\| = \|x\|$$

that is, f *preserves* norm.

Now we can consider

$$\begin{aligned} \|f(x) - f(y)\|^2 = \|x - y\|^2 \\ \implies \|f(x)\|^2 - 2\langle f(x), f(y)\rangle + \|f(y)\|^2 = \|x\| - 2\langle x, y\rangle + \|y\|^2 \\ \implies \langle f(x), f(y)\rangle = \langle x, y\rangle \end{aligned}$$

which means f *preserves* the inner product.

Using this, we wish to show f is *linear*

$$\begin{aligned} &\|f(x + \lambda y) - (f(x) + \lambda f(y))\|^2 \\ &= \|f(x + \lambda y)\|^2 - 2\langle f(x + \lambda y), (f(x) + \lambda f(y))\rangle + \|f(x) + \lambda f(y)\|^2 \\ &= \|f(x + \lambda y)\|^2 - 2\langle f(x + \lambda y), f(x)\rangle - 2\lambda\langle f(x + \lambda y), f(y)\rangle \\ &\quad + \|f(x)\|^2 - 2\lambda\langle f(x), f(y)\rangle + \lambda^2\|f(y)\|^2 \\ &= \|x + \lambda y\|^2 - 2\langle x + \lambda y, x\rangle - 2\lambda\langle x + \lambda y, y\rangle \\ &\quad + \|x\|^2 - 2\lambda\langle x, y\rangle + \lambda^2\|y\|^2 \\ &= \|x + \lambda y\|^2 - 2\|x + \lambda y\|^2 + \|x + \lambda y\|^2 \\ &= 0 \end{aligned}$$

Therefore

$$f \in O(\mathbb{R}^n)$$

Current note has 0 direct children and 0 total descendants.

- stretchy
  - $\mathbb{R}^n$
    - Isometric embedding of a subset of $\mathbb{R}^n$ into $\mathbb{R}^n$

And it has 4 siblings.

- stretchy
  - $\mathbb{R}^n$
    - Automorphisms of $\mathbb{R}^n$
    - $GL(n, \mathbb{R}^n) \curvearrowright \mathbb{R}^n$
    - Isometric embedding of a subset of $\mathbb{R}^n$ into $\mathbb{R}^n$
    - Dot product 2-norm on $\mathbb{R}^n$