Info

This note, that was found here, as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
was created on August 21, 2026 6:44:00 PM,
was last modified on September 4, 2026 2:06:24 AM, and
was exported on September 7, 2026 3:10:03 PM.

Schottky subgroups of $O^+(1,n)(\mathbb{R})$

Definition. Ultra-Schottky subgroups of $\mathrm{Moeb}(S^{n-1}) \cong O^+(1,n)$

Let $n, m \geq 1$. Consider $2m$ disjoint closed subsets

$$B_1, \ldots, B_m, B_1', \ldots, B_m' \subseteq S^{n-1}$$

each having a smooth boundary and homeomorphic to the closed $n-1$-ball and $\gamma_1, \ldots, \gamma_m \in \mathrm{Moeb}(S^{n-1})$ such that

$$\gamma_k(\mathring{B}_k) = S^{n-1} \setminus B_k'$$

Then

$$\Gamma := \langle \gamma_1, \ldots, \gamma_m \rangle$$

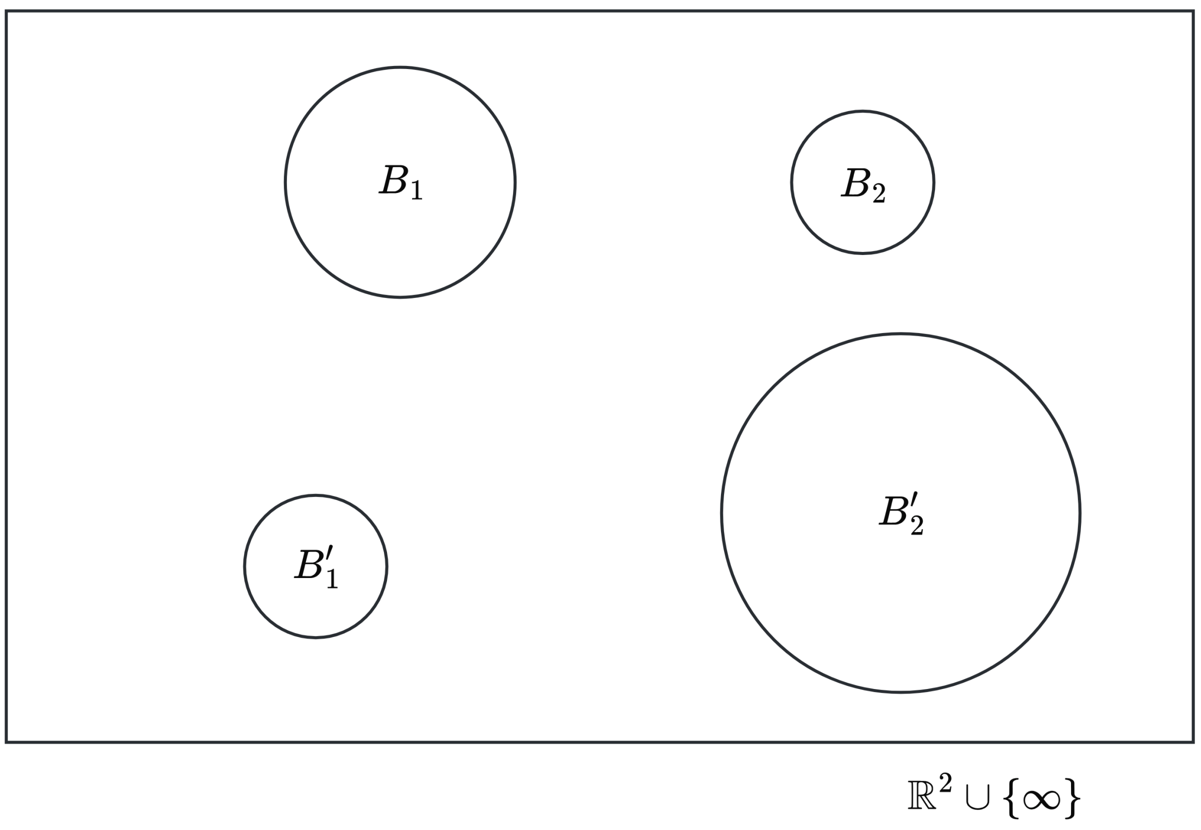
is a **ultra-Schottky subgroup** of $\mathrm{Moeb}(S^{n-1})$.

Fundamental domains of each side-pairing map

Let

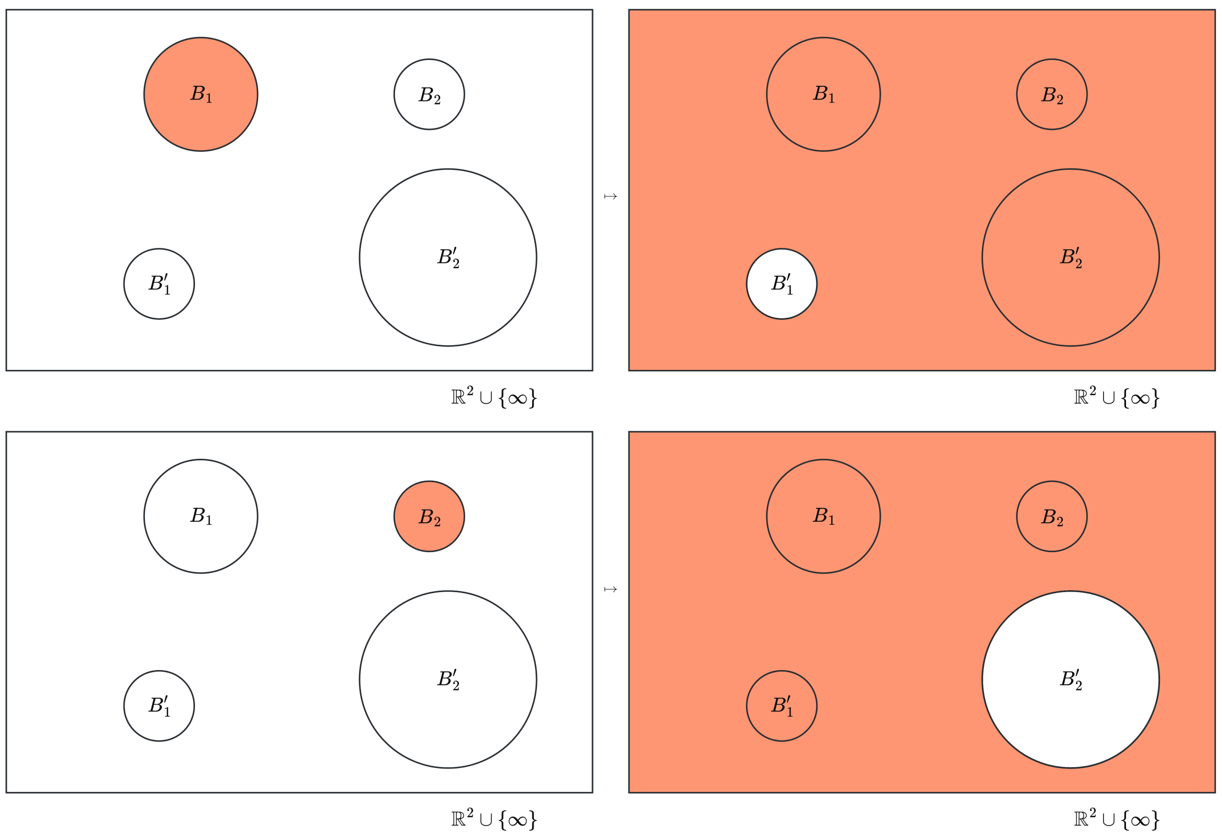
$$B_1, \ldots, B_m, B_1', \ldots, B_m'$$

be four disjoint closed balls in $\mathbb{R}^n \subset \mathbb{R}^n \cup \{\infty\}$.



Consider n Mobius maps $\gamma_1, \ldots, \gamma_n$ such that

$$\gamma_k(\mathring{B}_k) = (B_k')^{\mathsf{c}}$$



- This implies

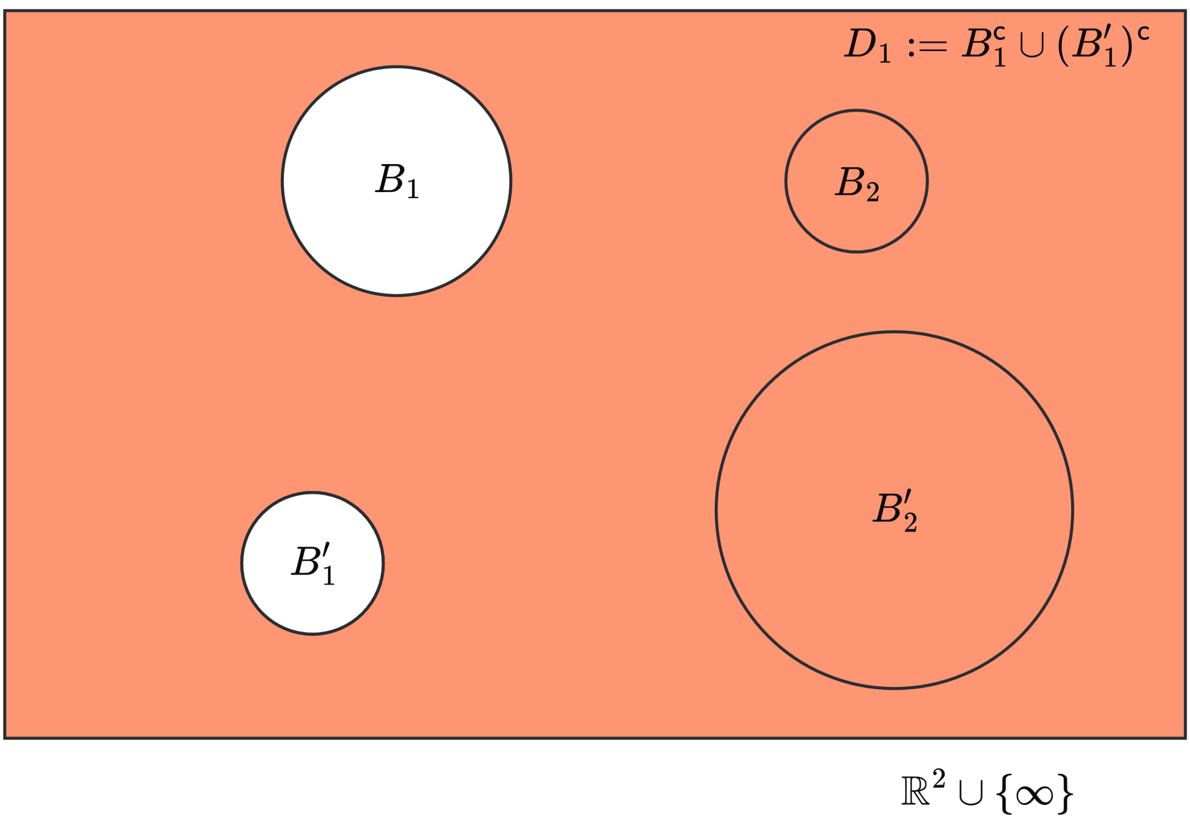
$$\gamma_k(B_k^{\mathsf{c}}) = \mathring{B}_k'$$

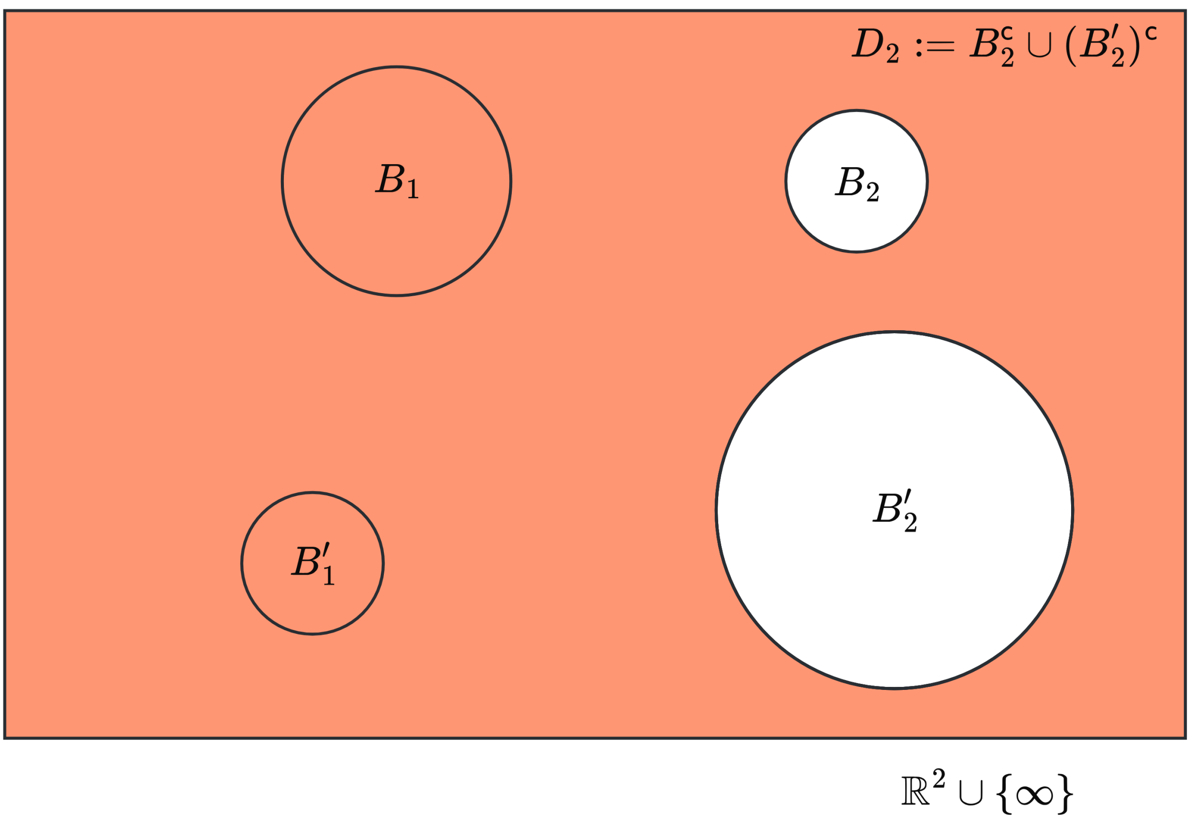
- In particular,

$$\begin{aligned} \mathring{B}_k' &\subset B_k^{\mathsf{c}} \\ \implies \gamma_k(\mathring{B}_k') &\subseteq \gamma_k(B_k^{\mathsf{c}}) \\ &= \mathring{B}_k' \end{aligned}$$

Definition.

$$D_k := B_k^{\mathsf{c}} \cap (B_k')^{\mathsf{c}}$$





In particular, we have

$$
\begin{aligned}
D_2^{\mathsf{c}} &\subseteq D_1 \\
D_1^{\mathsf{c}} &\subseteq D_2
\end{aligned}
$$

> **Proposition:** F_i is a fundamental domain of $\langle \gamma_i \rangle$

- **Base case:** From

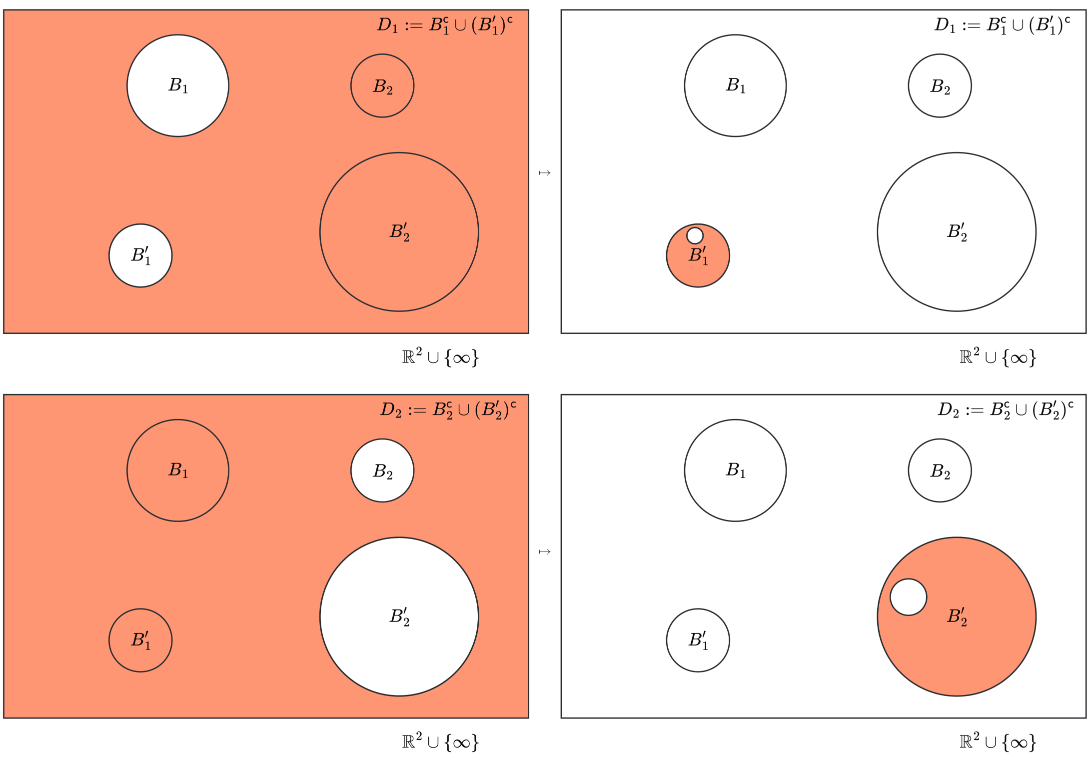
  > Consider n Mobius maps $\gamma_1, \ldots, \gamma_n$ such that
  >
  > $$\gamma_k(\mathring{B}_k) = (B_k')^{\mathsf{c}}$$

  and

  > - This implies
  >
  > $$\gamma_k(B_k^{\mathsf{c}}) = \mathring{B}_k'$$

  we have

  $$
  \begin{aligned}
  \gamma_k(D_k) &= \gamma_k(B_k^{\mathsf{c}} \cap (B_k')^{\mathsf{c}}) \\
  &\subset \mathring{B}_k' \\
  \implies \gamma_k(D_k) &\subset D_k^{\mathsf{c}}
  \end{aligned}
  $$

- **Induction hypothesis:** Let us assume for $n \geq 1$ we have

  $$\gamma_k^n(D_k) \subseteq \mathring{B}_k'$$

  So in paritcular, $\gamma_1^n(D_k) \subseteq D_k^{\mathsf{c}}$.
- **Induction step:** By

  > Definition.
  >
  > $$D_k := B_k^{\mathsf{c}} \cap (B_k')^{\mathsf{c}}$$

  and

- In particular,

$$
\begin{aligned}
& \mathring{B}_k' \subset B_k^{\mathsf{c}} \\
\implies & \gamma_k(\mathring{B}_k') \subseteq \gamma_k(B_k^{\mathsf{c}}) \\
& = \mathring{B}_k'
\end{aligned}
$$

we conclude

$$\gamma_k(\gamma_k^n(D_k)) \subseteq \gamma_k(\mathring{B}_k') \subseteq \mathring{B}_k'$$

Hence, $\gamma_1^n(D_1) \subseteq D_1^{\mathsf{c}}$.

- Therefore,

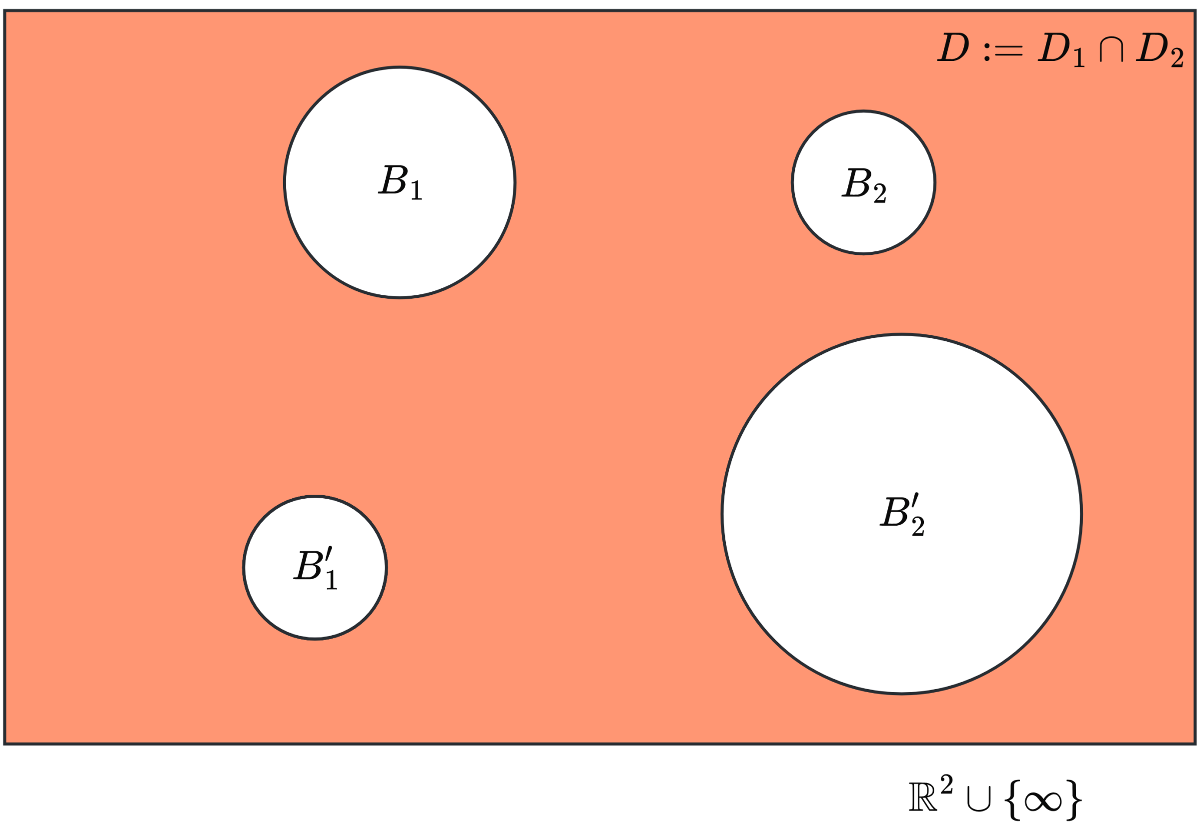
$$\gamma_1^n(D_1) \cap D_1 = \emptyset$$

- As γ_i is a hyperbolic-type transformation and $\mathrm{fix}(\gamma_i)$ consists of an attractive fixed point and repelling fixed point, we have

$$\bigcup_{n \in \mathbb{Z}} \gamma_i^n(F_i) = \mathbb{R}^n \cup \{\infty\} \setminus \mathrm{fix}(\gamma_i) = \Omega(\langle \gamma_i \rangle)$$

Fundamental domain of the group generated by side-pairing maps

Definition.
$$D := D_1 \cap \cdots \cap D_n$$



Proposition: $\Gamma := \langle \gamma_1, \ldots, \gamma_n \rangle$ is a free, discrete subgroup and

$$F := \overline{D}$$

becomes a fundamental domain for $\langle \gamma_1, \ldots, \gamma_n \rangle$.

- Let $x \in \mathbb{R}^n \cup \{\infty\} \setminus F$. As

$$F^{\mathsf{c}} = \mathring{B}_1 \sqcup \mathring{B}_1' \sqcup \mathring{B}_2 \sqcup \mathring{B}_2'$$

x must be in one of these balls.

  - Suppose $x \in \mathring{B}_1$. Because $\gamma_1(B_1^{\mathsf{c}}) = \mathring{B}_1'$ we have $\gamma_1^{-1}(x) \in B_1^{\mathsf{c}}$.
    - *either* $\gamma_1^{-1}(x) \in F$, then $x \in \gamma_1(F)$
    - *or,* $\gamma_1^{-1}(x)$ is in one of $\mathring{B}_1', \mathring{B}_2, \mathring{B}_2'$
      - then

$$\gamma_2^{-1}(\gamma_1^{-1}(x)) \in F$$

  - Repeating, we either have

$$x \in s_1 \ldots s_n F$$

or the procedure never terminates and we have

$$x \in U_{s_1} \cap s_1 U_{s_2} \cap \ldots$$

    - Then

$$\bigcap_{n \geq 1} s_1 \ldots s_{n-1} U_n = \{x\}$$

and

$$s_1 \ldots s_n(o) \xrightarrow{n \to \infty} x$$

    - Therefore, $x \in \Lambda(\Gamma) = \Omega(\Gamma)^{\mathsf{c}}$.
  - Thus

$$\Omega(\Gamma) = \bigcup_{\gamma \in \Gamma} \gamma(F)$$

Current note has 0 direct children and 0 total descendants.

- stem
  - O+ R1 n
    - Discrete subgroups of $O^+(1,n)$ $\leftrightarrow$ hyperbolic n-manifolds/orbifolds
      - Schottky
        - Schottky subgroups of $O^+(1,n)$

And it has 2 siblings.

- stem
  - O+ R1 n
    - Discrete subgroups of $O^+(1,n)$ $\leftrightarrow$ hyperbolic n-manifolds/orbifolds
      - Schottky
        - Classical Schottky subgroups of $O^+(1,n)$
        - Schottky subgroups of $O^+(1,n)$