Info

This note, that was found here, as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
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Exact sequences in singular chain complex from inclusions

Let $U, V \subseteq X$ of a topological space. We have the following inclusions

```
\usepackage{tikz-cd}\usepackage{amsfonts}\usepackage{amsmath}
\begin{document}

%
https://q.uiver.app/#q=WzAsNSxbMCwxLCJVIFxcY2FwIFYiXSxbMSwwLCJVIl0sWzEsMi
wiViJdLFszLDEsIlUgXFxjdXAgViJdLFsyLDEsIlUgXFxjb3Byb2QgViJdLFswLDEsIlxcaW9
0YV97VVxcY2FwIFZ9IiwwLHsic3R5bGUiOnsidGFpbCI6eyJuYW1lIjoiaG9vayIsInNpZGUi
OiJ0b3AifX19XSxbMCwyLCJcXGlvdGFfe1VcXGNhcCBWfSIsMix7InN0eWxlIjp7InRhaWwiO
nsibmFtZSI6Imhvb2siLCJzaWRlIjoidG9wIn19fV0sWzEsNCwiXFxpb3RhX3tVfSIsMCx7In
N0eWxlIjp7InRhaWwiOnsibmFtZSI6Imhvb2siLCJzaWRlIjoidG9wIn19fV0sIzIsNCwiXFx
pb3RhX3tWfSIsMix7InN0eWxlIjp7InRhaWwiOnsibmFtZSI6Imhvb2siLCJzaWRlIjoidG9w
In19fV0sWzQsMywiXFxwaSJdXQ==
\begin{tikzcd}
    & U \\
    {U \cap V} && {U \coprod V} & {U \cup V} \\
    & V
    \arrow["{\iota_{U\cap V}}", hook, from=2-1, to=1-2]
    \arrow["{\iota_{U\cap V}}"', hook, from=2-1, to=3-2]
    \arrow["{\iota_{U}}", hook, from=1-2, to=2-3]
    \arrow["{\iota_{V}}"', hook, from=3-2, to=2-3]
    \arrow["\pi", from=2-3, to=2-4]
\end{tikzcd}

\end{document}
```

which on applying sing_* functor gives

```
\usepackage{tikz-cd}\usepackage{amsfonts}\usepackage{amsmath}
\begin{document}

 %
https://q.uiver.app/#q=WzAsNSxbMCwxLCJcXG1hdGhybXtzaW5nfV97Kn0oVSBcXGNhcC
BWKSJdLFsxLDAsIlxcbWF0aHJte3Npbmd9X3sqfShVKSJdLFsxLDIsIlxcbWF0aHJte3Npbmd
9X3sqfShWKSJdLFszLDEsIlxcbWF0aHJte3Npbmd9X3sqfShVIFxcY3VwIFYpIl0sWzIsMSwi
XFxtYXRocm17c2luZ31feyp9KFUpIFxcb3BsdXMgXFxtYXRocm17c2luZ31feyp9KFYpIl0sW
zAsMSwiXFxpb3RhX3tVXFxjYXAgVip9IiwwLHsic3R5bGUiOnsidGFpbCI6eyJuYW1lIjoiaG
9vayIsInNpZGUiOiJ0b3AifX19XSxbMSw0LCJcXGlvdGFfe1UqfSIsMCx7InN0eWxlIjp7InR
haWwiOnsibmFtZSI6Imhvb2siLCJzaWRlIjoidG9wIn19fV0sWzQsMywiXFxpb3RhX3tWKn0i
LDIseyJzdHlsZSI6eyJ0YWlsIjp7Im5hbWUiOiJob29rIiwic2lkZSI6InRvcCJ9fX1dLFs0L
DMsIlxccGlfKiJdLFswLDEsIlxcaW90YV97VVxcY2FwIFYqfSIsMix7InN0eWxlIjp7InRhaW
wiOnsibmFtZSI6Imhvb2siLCJzaWRlIjoidG9wIn19fV1d
 \begin{tikzcd}
    & {\mathrm{sing}_{*}(U)} \\
    {\mathrm{sing}_{*}(U \cap V)} && {\mathrm{sing}_{*}(U) \oplus
\mathrm{sing}_{*}(V)} & {\mathrm{sing}_{*}(U \cup V)} \\
    & {\mathrm{sing}_{*}(V)}
    \arrow["{\iota_{U\cap V*}}", hook, from=2-1, to=1-2]
    \arrow["{\iota_{U*}}", hook, from=1-2, to=2-3]
    \arrow["{\iota_{V*}}"', hook, from=3-2, to=2-3]
    \arrow["{\pi_*}", from=2-3, to=2-4]
    \arrow["{\iota_{U\cap V*}}"', hook, from=2-1, to=3-2]
 \end{tikzcd}

 \end{document}
```

Now we define the short sequence

```
\usepackage{tikz-cd}\usepackage{amsfonts}\usepackage{amsmath}
\begin{document}

%
https://q.uiver.app/#q=WzAsOSxbMSwwLCJcXG1hdGhybXtzaW5nfV97Kn0oVSBcXGNhcC
BWKSJdLFs0LDAsIlxcbWF0aHJte3Npbmd9X3sqfShVIFxcY3VwIFYpIl0sWzMsMCwiXFxtYXR
ocm17c2luZ31feyp9KFUpIFxcb3BsdXMgXFxtYXRocm17c2luZ31feyp9KFYpIl0sWzAsMCwi
MCJdLFs1LDAsIjAiXSxbMSwxLCJ4Il0sWzMsMSwiKHgsLXgpIl0sWzQsMSwiKHgseSkiXSxbN
CwyLCJ4K3kiXSxbMiwxLCJcXHBpXyoiXSxbMCwyLCJcXGlvdGFfe1UqfSBcXHNxY3VwLVxcaW
90YV97VSp9Il0sWzMsMSwiF0sWzEsNF0sWzcsOCwiIiwwLHsic3R5bGUiOnsidGFpbCI6eyJ
lIjoibWFwcyB0byJ9fX1dLFs1LDYsIiIsMCx7InN0eWxlIjp7InRhaWwiOnsibmFtZSI6Im1h
cHMgdG8ifX19XV0=
\begin{tikzcd}
    0 & {\mathrm{sing}_{*}(U \cap V)} && {\mathrm{sing}_{*}(U) \oplus
\mathrm{sing}_{*}(V)} & {\mathrm{sing}_{*}(U \cup V)} & 0 \\
    & x && {(x,-x)} \\
    &&& {(x,y)} & {x+y}
```

```
    \arrow["{\pi_*}", from=1-4, to=1-5]
    \arrow["{\iota_{U*} \times-\iota_{U*}}", from=1-2, to=1-4]
    \arrow[from=1-1, to=1-2]
    \arrow[from=1-5, to=1-6]
    \arrow[maps to, from=3-4, to=3-5]
    \arrow[maps to, from=2-2, to=2-4]
\end{tikzcd}

\end{document}
```

- At the first position,

$$x \mapsto (x, -x)$$

so $(x, -x) = 0 \iff x = 0$ implies

$$\ker(\iota_{U*} \times -\iota_{V*}) = 0$$

- At the second position any element of the image $(x, -x)$ maps to 0. Now, if (x, y) maps to 0, then

$$x = -y$$

where $x \in \mathrm{sing}_*(U)$ and $y \in \mathrm{sing}_*(V)$ which means the chain x lines in $U \cap V$ hence, $x = -y \in \mathrm{sing}_*(U \cap V)$. Hence, the sequence is exact at this position.
- At the third position, any $z \in \mathrm{sing}_*(U \cup V)$, **it is not exact??**

We replace the space at the third position with the subspace which is the image of the map π_*

```
\usepackage{tikz-cd}\usepackage{amsfonts}\usepackage{amsmath}
\begin{document}

%
https://q.uiver.app/#q=WzAsOSxbMSwwLCJcXG1hdGhybXtzaW5nfV97Kn0oVSBcXGNhcC
BWKSJdLFs0LDAsIlxcbWF0aHJte3Npbmd9X3sqfShVKStcXG1hdGhybXtzaW5nfV97Kn0oVik
iXSxbMywwLCJcXG1hdGhybXtzaW5nfV97Kn0oVSkgXFxvcGx1cyBcXG1hdGhybXtzaW5nfV97
Kn0oVikiXSxbMCwwLCIwIl0sWzUsMCwiMCJdLFsxLDEsIngiXSxbMywxLCIoeCwteCkiXSxbM
ywyLCIoeCx5KSJdLFs0LDIsIngreSJdLFsyLDEsIlxccGlfKiJdLFswLDIsIlxcaW90YV97VS
p9IFxcc3FjdXAtXFxpb3RhX3tVKn0iXSxbMywwXSxbMSw0XSxbNyw4LCIiLDAseyJzdHlsZSI
6eyJ0YWlsIjp7Im5hbWUiOiJtYXBzIHRvIn19fV0sWzUsNiwiIiwwLHsic3R5bGUiOnsidGFp
bCI6eyJuYW1lIjoibWFwcyB0byJ9fX1dXQ==
\begin{tikzcd}
    0 & {\mathrm{sing}_{*}(U \cap V)} && {\mathrm{sing}_{*}(U) \oplus
\mathrm{sing}_{*}(V)} & {\mathrm{sing}_{*}(U)+\mathrm{sing}_{*}(V)} & 0
\\
    & x && {(x,-x)} \\
    &&& {(x,y)} & {x+y}
    \arrow["{\pi_*}", from=1-4, to=1-5]
    \arrow["{\iota_{U*} \times-\iota_{U*}}", from=1-2, to=1-4]
    \arrow[from=1-1, to=1-2]
    \arrow[from=1-5, to=1-6]
    \arrow[maps to, from=3-4, to=3-5]
    \arrow[maps to, from=2-2, to=2-4]
\end{tikzcd}

\end{document}
```

now this sequence is a *short exact sequence*!

Current note has 0 direct children and 0 total descendants.

- structures based on sets
    - Topological spaces
        - complex sing
            - Singular chain complex of a topological space over $\mathbb{Z}$
                - Singular chain complex over $\mathbb{Z}$ functor on topological spaces
                    - Exact sequences in singular chain complex from inclusions

And it has 2 siblings.

- structures based on sets
    - Topological spaces
        - complex sing
            - Singular chain complex of a topological space over $\mathbb{Z}$
                - Singular chain complex over $\mathbb{Z}$ functor on topological spaces
                    - Homotopy invariance of singular homology functor on topological spaces
                    - Exact sequences in singular chain complex from inclusions