ⓘ Info

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Harmonic functions on Riemannian manifold

Definition. Let (M, g) be a Riemannian manifold. Then $u \in \mathcal{C}^\infty(M)$ is a **harmonic** function if

$$\Delta u \equiv 0$$

harmonic functions on closed manifold are constants

Let (M, g) be a connected orientable closed manifold.

From

> **Proposition:**
>
> $$\Delta_{\mathrm{g}}(fh) = f(\Delta_{\mathrm{g}} h) + 2\left\langle \mathrm{grad}_{\mathrm{g}}(f), \mathrm{grad}_{\mathrm{g}}(h)\right\rangle + h(\Delta_{\mathrm{g}} f)$$

and Stokes theorem we have

$$\cancelto{0}{\int_M \Delta_{\mathrm{g}}(fh)\lambda_{\mathrm{g}}} = \int_M (f(\Delta_{\mathrm{g}} h) + 2\left\langle \mathrm{grad}_{\mathrm{g}}(f), \mathrm{grad}_{\mathrm{g}}(h)\right\rangle + h(\Delta_{\mathrm{g}} f))\lambda_{\mathrm{g}}$$

Let $h = f/2$ and $\Delta f \equiv 0$ then

$$\begin{aligned} 0 &= \int_M (f\,\cancelto{0}{(\Delta_{\mathrm{g}} h)} + 2\left\langle \mathrm{grad}_{\mathrm{g}}(f), \mathrm{grad}_{\mathrm{g}}(h)\right\rangle + h\,\cancelto{0}{(\Delta_{\mathrm{g}} f)})\lambda_{\mathrm{g}} \\ &= \int_M \|\mathrm{grad}(f)\|^2 \lambda_{\mathrm{g}} \\ \implies & \mathrm{grad}(f) \equiv 0 \\ & \implies f \text{ is constant} \end{aligned}$$

Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - manifolds
    - $\mathbb{R}$-manifold semi-Riemannian
      - Laplacian on a Riemannian manifold
        - Harmonic functions on Riemannian manifold

And it has 2 siblings.

- structures based on sets
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    - $\mathbb{R}$-manifold semi-Riemannian
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        - Conjugating the Laplacian
        - Harmonic functions on Riemannian manifold