Info

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Canonical bijection of T_pM with equivalence classes of velocities

Definition. Set of smooth curves

The set of all **smooth curves** is the set $\mathcal{C}^\infty(I, U)$ where $I \subseteq \mathbb{R}, U \subseteq M$. We define the set of all curves with the naturally desired domain,

$$\gamma : (-\epsilon, \epsilon) \to U \subseteq M$$

for some $\epsilon > 0$ as

$$\mathrm{Curves}^\infty_\epsilon(p \in U) := \{\gamma \in \mathcal{C}^\infty((-\epsilon, \epsilon), U) : \gamma(0) = p\}$$

equivalence classes of velocities

For a open set U on a n-manifold, and a chart (U, φ)

$$\mathrm{Curves}^\infty(p \in U) \to \mathbb{R}^n$$

such that

$$\gamma \mapsto (\varphi \circ \gamma)'(0)$$

this is a surjective map. Lets denote the map by the ugly $(\varphi \circ -)'(0)$. This map shall partition the domain

$$\mathrm{Curves}^\infty(p \in U)/(\varphi \circ -)'(0)$$

where the equivalence relation is

$$\gamma_1 \sim \gamma_2 \iff (\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0)$$

The $\mathrm{Curves}^\infty(p \in U)/(\varphi \circ -)'(0)$ partition does not depend on the chart φ.

Definition. The equivalence classes of velocity

Denote the quotient set of the equivalence classes of velocity by

$$\mathrm{Curves}^\infty(p \in U)/(-)'(0)$$

and the equivalence classes by

$$[\gamma'(0)]$$

because it does not depend on chart.

canonical bijection

We have

$$\begin{tikzcd}
T_pM \arrow[r, "(-\circ\gamma)'(0)\mapsto(\varphi\circ\gamma)'(0)"] & \mathbb{R}^{\dim M} \\
\mathrm{Curves}^\infty(p \in U) \arrow[ur, "\gamma \mapsto (\varphi \circ \gamma)'(0)"'] &
\end{tikzcd}$$

From

For a n-manifold M, map

$$T_pM \to \mathbb{R}^n$$
$$(- \circ \gamma)'(0) \mapsto (\varphi \circ \gamma)'(0)$$

is an $\mathbb{R}$-linear isomorphism for any local chart φ at p. Hence,

$$\dim(T_pM) = \dim(M).$$

we know the operator $(- \circ \gamma)'(0) \in T_pM$ is **same** for two curves in $[\gamma'(0)]$ which is

$$(- \circ \gamma_1)'(0) = (- \circ \gamma_2)'(0) \iff (\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0)$$

Definition. The **canonical bijection** $T_pM \to \mathrm{Curves}^\infty(p \in U)/(-)'(0)$ is defined as

$$(- \circ \gamma)'(0) \leftrightarrow [\gamma'(0)]$$

This the diagram

$$\begin{tikzcd}
T_pM \arrow[r, "(-\circ\gamma)'(0)\mapsto(\varphi\circ\gamma)'(0)"] & \mathbb{R}^{\dim M} \\
\mathrm{Curves}^\infty(p \in U) \arrow[u, "(- \circ \gamma)'(0) \mapsto {[\gamma'(0)]}"] \arrow[ur, "\gamma \mapsto (\varphi \circ \gamma)'(0)"'] &
\end{tikzcd}$$

commutes and each arrow is an isomorphism.

Current note has 0 direct children and 0 total descendants.

- structures based on sets
    - manifolds
        - smooth $\mathbb{R}$-manifold
            - Tangents on a smooth manifold
                - Tangent space of a smooth manifold
                    - Canonical bijection of T_pM with equivalence classes of velocities

And it has 8 siblings.

- structures based on sets
    - manifolds
        - smooth $\mathbb{R}$-manifold
            - Tangents on a smooth manifold
                - Tangent space of a smooth manifold
                    - Canonical bijection of T_pM with equivalence classes of velocities
                    - Tangents on smooth manifolds as derivations $\mathcal{C}^\infty(M) \to \mathbb{R}$
                    - local basis of a tangent space of a smooth manifold
                    - Tangent bundle TM of a smooth manifold M
                    - Cotangent bundle T^*M on a smooth manifold M
                    - Comparison of constructions with tangent vectors
                    - Tangent frames on smooth manifolds
                    - Tangent vector fields on a smooth manifold