Info

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Global holomorphic functions

> Quote
>
> **Riemann:** What is the *domain of definition* of an holomorphic function?

Let the union of all germs of holomorphic functions ("*Etale space*") be

$$\mathscr{O} := \left\{ (a, f) : a \in \mathbb{C},\ f(z) = \sum_{n \geq 0} c_n (z-a)^n \right\}$$

and we want a topology on this such that

$$\begin{aligned} \mathscr{O} &\to \mathbb{C} \\ (a, f) &\mapsto a \end{aligned}$$

is a local homeomorphism.

We define

$$\begin{aligned} s_f : a + r_a D &\to \mathscr{O} \\ b &\mapsto (b, T_b f) \end{aligned}$$

where given a $(a, f) \in \mathscr{O}$ we have

$$(T_b f)(z) := \sum_{n \geq 0} f^n(b) \frac{z-b}{n!}$$

Then the images are

$$s_f(a + r_a D) = \{(b, g) \mid\}$$

> **Proposition:**
>
> $$\{s_f(a + r_a D) : (a, f) \in \mathscr{O}\}$$
>
> becomes a sub-basis for a topology on $\mathscr{O}$.

Now

$$\mathscr{O}$$

is the space of all global holomorphic functions with (domain points from $\mathbb{C}$ and valued in $\mathbb{C}$).

On this topological space we have two maps

- the projection π onto $\mathbb{C}$
- the *evaluation* of the function

$$\begin{tikzcd} & \substack{\mathscr{O} \\ (a,f)} \arrow[r, "\mathrm{ev}"] \arrow[d, "\pi"] & \substack{\mathbb{C} \\ f(a)} \\ a + r_a D \arrow[ur, "s_f"] \arrow[r] & \substack{\mathbb{C} \\ a} & \end{tikzcd}$$

Definition 1. We define the *derivative* d, a map $d\colon \mathcal{O} \to \mathcal{O}$ as follows: If $f_a \in \mathcal{O}_a$ and (U, f) is a representative of f_a, we define $d(f_a) = df_a$ to be the germ at a of (U, f') where $f' = df/dz$ is the derivative of f.

Proposition 1. *$d\colon \mathcal{O} \to \mathcal{O}$ is a covering map.*

Proof. Let $f_a \in \mathcal{O}$ and let (U, f) be a representative of f_a. Let D be a disc centered at a such that $D \subset U$.

Let F be a primitive of f on D, and let $\mathcal{D} = N(D, f)$. For any $c \in \mathbb{C}$, let $U_c = N(D, F + c)$. We claim that $d^{-1}(\mathcal{D}) = \bigcup_{c \in \mathbb{C}} U_c$.

To prove this, let $z \in D$, $g_z \in \mathcal{O}_z$ and $dg_z = f_z$. Let (W, g) be a representative of g_z where W is a connected neighborhood of z, $W \subset D$. Then $g' = f$ in a neighborhood of z, hence $g' = f$ on W so that $(d/dz)(g - F) = 0$ on W. Hence, for some $c \in \mathbb{C}$, $g = F + c$ on W and $g_z \in U_c$. Conversely, we trivially have $dU_c = N(D, f) = \mathcal{D}$.

We now check that $d|U_c$ is a homeomorphism onto $\mathcal{D}$ for any $c \in \mathbb{C}$. We have only to check that $d|U_c$ is injective, which is obvious since d takes distinct elements of U_c to germs at different points of D. Since, finally, the U_c with distinct values of c are mutually disjoint, $\mathcal{D}$ is evenly covered by d.

We shall now show that there is a close connection between integration along curves and the lifting of curves relative to the derivative $d\colon \mathcal{O} \to \mathcal{O}$.

connected components

A connected component of $\mathscr{O}$ is a "Riemann surface" S. There are two functions on it

$$S \to \mathbb{C}; (a, f) \mapsto a$$

$$S \to \mathbb{C}; (a, f) \mapsto f(a)$$

- a $(0, f)$ whose connected component only contains disk?

constructing global holomorphic functions

- The complex logarithm
- Radicals $z^{1/k}$
- Inverse of a holomorphic function $f : U \to \mathbb{C}$
- Primitive if a 1-form $f(z)\mathrm{d}z$ on $U \subseteq \mathbb{C}$

- Radical $\sqrt[k]{f}$ of a holomorphic function $f: U \to \mathbb{C}$

adding the missing points: meromorphic functions on $\mathbb{C}P^1$

The corresponding space for meromorphic functions on $\mathbb{C}P^1$ is

$$\begin{aligned}\mathscr{M} := &\left\{(a, f) \middle| f = \sum_{n=-N}^{\infty} c_n (z-a)^n, \limsup_n |c_n|^{1/n} < \infty\right\} \\ &\cup \left\{(\infty, f) \middle| f(z) = \sum_{n=-N}^{\infty} c_n \frac{1}{z^n}, \limsup_n |c_n|^{1/n} < \infty\right\}\end{aligned}$$

So

$$\mathscr{O} \subset \mathscr{M}$$

but its larger because

$$\left(0, \frac{1}{z}\right) \in \mathscr{M}$$

We extend the topology on $\mathscr{O}$ to a topology on $\mathscr{M}$.

We have the extended maps

$$\begin{array}{ccc} \mathscr{M} & \xrightarrow{\text{ev}} & \mathbb{C}P^1 \\ (a, f) & & f(a) \\ {\scriptstyle\pi}\downarrow & & \\ \mathbb{C}P^1 & & \\ a & & \end{array}$$

where

> **Proposition:** π is a local homeomorphism.

(check for the points at infinity...)

adding the missing points: finite ramifications

Consider rD^* and its inverse image in $\mathscr{M}$. There are many connected components of the pre-image. We consider a component $E \subset \mathscr{M}$ such that

$$\pi : E^* \to rD^*$$

is **proper**.

These exist.

And now by

> **Proposition:**
> 11-9. Show that a proper local homeomorphism between connected, locally path-connected, and compactly generated Hausdorff spaces is a covering map.
> 11-10. Show that a covering map is proper if and only if it is finite-sheeted.

we know

$$\pi : E^* \to rD^*$$

is a finite, connected, covering map of the punctured disk!

They are classified by finite subgroups $\pi_1(D^*) \cong \mathbb{Z}$, and they are of the form:

$$D^* \to D^*, z \mapsto z^n$$

or

$$\mathscr{O}[z^{1/n}] \times_{\mathbb{C}^*} D^* = D^*_{1/n} =: \pi^{-1}(D^*) \xrightarrow{\pi} D^*$$

which gives

$$\begin{tikzcd} & E^* \arrow[r] \arrow[dl, "f"'] & D^* \\ \mathbb{C}P^1 & D^*_{1/n} \arrow[l] \arrow[u] \arrow[r, hook] \arrow[ur] & D_{1/n} \end{tikzcd}$$

$$\prod_i (w - f \circ \gamma^i) = w^n + a_1 w^{n-1} + \cdots + a_n$$

then these a_i are holomorphic functions well defined on D^* which has a removable singularity at 0, so they extend onto D. Now by

(Newton-Puiseux series)

[1]

we can solve for f on $D_{1/n}$.

Now we add these points

$$(0, -)$$

...

and we construct

$$\overline{\mathscr{M}}$$

adds onto $\mathscr{M}$ the "punctures" to the punctured disks in $\mathscr{M}$ that occurs as finite covers of punctured disks in $\mathbb{C}P^1$.

The components of $\overline{\mathscr{M}}$ solves the problem of finding the maximal domain of definition of a meromorphic function on $\mathbb{C}P^1$.

Current note has 0 direct children and 0 total descendants.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $\Gamma\backslash\mathbb{R}^n, \Gamma\backslash\mathbb{C}^n$
    - Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - Global holomorphic functions

And it has 19 siblings.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $\Gamma\backslash\mathbb{R}^n, \Gamma\backslash\mathbb{C}^n$
    - Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - Approximation of holomorphic functions on $\mathbb{C}$ by rational functions
      - Embedding unit disk D into $\mathbb{C}^3$
      - Factorization of holomorphic functions on $\mathbb{C}$
      - Global holomorphic functions
      - Equivalent descriptions of holomorphic functions
      - Meromorphic functions on the complex plane
      - Holomorphic functions meromorphic at infinity
      - Modular forms
      - Reconstructing meromorphic functions from stalks
      - Extending holomorphic functions by reflections
      - Rotational symmetrization of holomorphic functions
      - Sheaf of holomorphic functions on $\mathbb{C}$
      - $\mathscr{O}(U)$
      - $\mathscr{O}(\mathbb{C})$
      - $\mathscr{O}(D)$
      - $\mathscr{O}^p(H^2_{\mathrm{U}})$
      - $\mathscr{O} \cap L^p$
      - $\mathscr{O}(S^1)$
      - Zeros and singularities of holomorphic functions

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1. Walker book ↩