ⓘ Info

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Metric cube complexes

Definition. Cubes

A **cube** is the metric space

$$(C, d_C) := ([0,1]^n, d_{\mathbb{R}^n})$$

- A **face** of **codimension 1** of C is given by

$$F_{i,j} := C \cap \{x_i = j\}$$

  for $j \in \{0,1\}, i \in \{1,\ldots,n\}$
- All other **proper faces** of C are non-empty intersections of codimension 1 faces.
- A face (which is not a proper face) of C is C itself

A point $x \in C$ is an **inner point** if x is not contained in any proper face of C.

Definition. Cubical complexes

Let C, C' be two cubes with faces $F \subseteq C, F' \subseteq C'$. A **gluing** of C and C' is an *isometry* $\phi : (F, d_C) \to (F', d_{C'})$.
Suppose $\mathcal{C}$ is a set of cubes and $\mathcal{S}$ be a family of gluings of pairs of elements in $\mathcal{C}$, that is,

$$\begin{aligned} \forall C \in \mathcal{C} \exists n_C \in \mathbb{N} \quad &\text{st} \quad (C, d_C) = ([0,1]^{n_C}, d_C) \\ \forall \phi \in \mathcal{S} \exists C, C' \in \mathcal{S}, & \\ F, F'(\text{faces}) \subseteq C, C' \quad &\text{st} \quad \phi : (F, d_C) \xrightarrow{\text{isometry}} (F', d_{C'}) \end{aligned}$$

Assume $(\mathcal{C}, \mathcal{S})$ satisfies the following conditions

- no cube is glued to itself
- for all $C \neq C' \in \mathcal{S}$ there is at most one gluing of C and C'

then this pair defines the **cubical complex**

$$X := \frac{\coprod_{C \in \mathcal{C}} C}{x \sim \phi(x) \iff \mathcal{S} \ni \phi : F \to F', x \in F}$$

with projection

$$\pi : \coprod_{C \in \mathcal{C}} C \to X$$

and metric

$$d_X := \overline{\pi^* \left(\coprod_{C \in \mathcal{C}} d_C \right)}$$

A cubical complex is

- **finite dimensional** if there exists an upper bound on the dimension of cubes in $\mathcal{C}$

$$\sup \{n_C | C \in \mathcal{C}\} < \infty$$

- **locally finite** if no point of X is contained in infinitely many cubes

[1]

> **Proposition:** A cubical complex is a complete geodesic metric space if it is either finite dimensional or locally finite.

> **Proposition:** A cube complex has two topologies: one given by the metric d_X and the quotient topology:
>
> $$\mathfrak{T}_{d_X} \subseteq \mathfrak{T}_\pi$$
>
> The two topologies are equal $\iff$ X is locally finite.

(Gromov's link condition) A finite dimensional cubical complex is locally CAT(0) $\iff$ all its vertex links are flag simplicial complexes.

Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - Metric spaces
    - Metric simplicial complexes
      - Metric cube complexes

And it has 1 siblings.

- structures based on sets
  - Metric spaces
    - Metric simplicial complexes
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1. https://arxiv.org/pdf/1910.06815 ↩