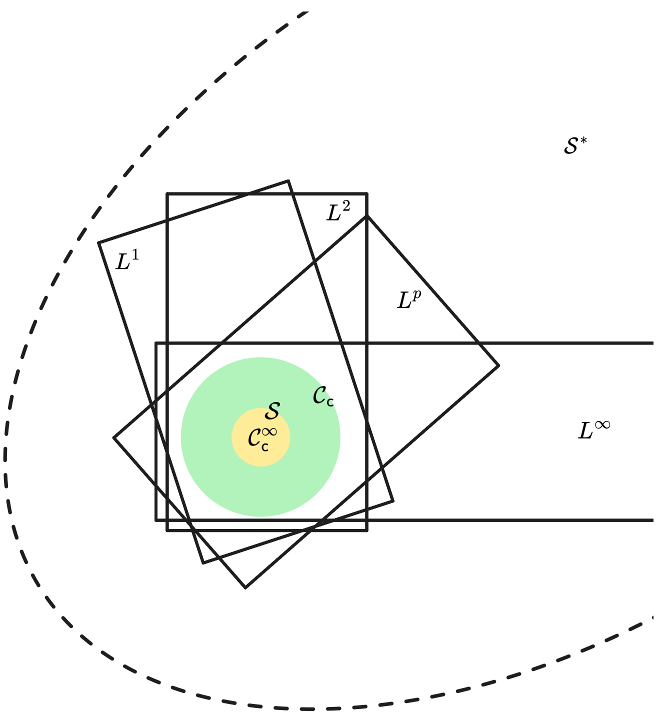
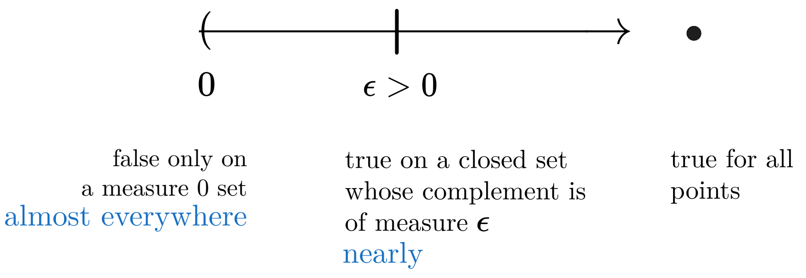
# Lebesgue measurable subsets of and functions on $\mathbb{R}^n, T^n, S^n$




New notions on equality and properties!

- Some property p is true for elements of a measurable set $E$ *almost everywhere* if it is only false on a measure 0 set.
- Some property p is *nearly true* for elements of a measurable set $E$ if for all $\epsilon > 0$ there exists a closed set $F_\epsilon \subseteq E$ such that $m(E \setminus F_\epsilon) \leq \epsilon$ and it is true on $F_\epsilon$.

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Littlewood's three principles about measurable sets and functions say

- Every set is *nearly(?)* a finite union of intervals.
- Every convergent sequence of measurable functions is *nearly* uniformly convergent.

  > **(Ergorov)** Let $(f_n)$ be a sequence of measurable function on $E \subseteq \mathbb{R}^d$ with $m(E) < \infty$ and
  >
  > $$f_n \to f \text{ aeon } E$$
  >
  > Then for any given $\epsilon > 0$ we can find a closed set $A_\epsilon \subset E$ such that $m(E - A_\epsilon) \leq \epsilon$ and
  >
  > $$f_n \xrightarrow{\text{uniformly}} f \text{ on } E$$

- Every measurable function is *nearly* continuous.

  > **(Lusin)** Let $f$ be measurable function on $E \subseteq \mathbb{R}^d$ with $m(E) < \infty$. Then for every $\epsilon > 0$ there exists a closed set $F_\epsilon \subset E$ such that $m(E - F_\epsilon) \leq \epsilon$ so that
  >
  > $$f\Big|_{F_\epsilon}$$
  >
  > is continuous.

| subsets | functions |
|---|---|
| | **Definition.** Measurable functions on $\mathbb{R}^n$<br>Let $E \subseteq \mathbb{R}^n$ be a measurable subset. A function $f : E \to \mathbb{R}$ is **measurable** if $$f^{-1}(-\infty, a) = \{f < a\}$$ is measurable for all $a \in \mathbb{R}$. |
| **(Lebesgue density theorem)** For almost every $x \in E$, we have $$\text{density}_E(x) = 1$$ | **(Differentiation of the Lebesgue integral on balls)** If $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ then $$\lim_{r \to 0}(A_r f)(x) = f(x) \text{ aeon } x \in \mathbb{R}^d$$ |

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Current note has 15 direct children and 16 total descendants.

- Decomposition of $f \in L^p$ into bounded and unbounded parts
- Calderon-Zygmund decomposition
- Lebesgue density of measurable sets
- Functions with uniformly bounded mean oscillations on dyadic cubes
- Volterra operator $\int_{[0,-]} : L^1_{\mathrm{loc}}[0,1) \to \mathcal{C}[0,1)$
  - Volterra operator $\int_{[0,-]} : L^p[a,b] \to L^p[a,b]$
- Measurable functions on $\mathbb{R}^n$
- $(\epsilon, n)$-measurable function
- Integrability and integral of measurable functions on $\mathbb{R}^n$
- Hardy-Littlewood maximal functions of $L^1_{\mathrm{loc}}(\mathbb{R}^n)$
- Lebesgue averaging and differentiation
- Integrals of a monotonically converging sequence of functions
- Lebesgue integral from measure of undergraph
- $L^{p,q}$
- $\mathrm{E}([0,1]) = \mathrm{E}(0,1), \mathrm{E}([-\pi,\pi]) = \mathrm{E}(S^1)$

And it has 39 siblings.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $\Gamma\backslash^{\mathbb{R}^n}, \Gamma\backslash^{\mathbb{C}^n}$
    - Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
    - Open subsets of $\mathbb{R}^n$ with $\mathcal{C}^2$ boundary
    - Complex ODEs in $\mathbb{C}$
    - Circle packing on $\mathbb{R}^2$
    - Circle packing converges to the Riemann biholomorphism
    - Bounded connected open subsets of $\mathbb{C}^n$
    - Connected circular open subsets of $\mathbb{C}^n$
    - Continuous functions on $\mathbb{R}^d$
    - Dyadic cubes
    - Curves
    - Differentiable functions
    - Differential forms on $\mathbb{R}^n$
    - Fourier-Wigner transform
    - Harmonic functions composed with conformal maps
    - Hilbert transform
    - Fourier-Cauchy-Poisson correspondence of holomorphic and harmonic functions on the unit disk and their boundary values
    - Holomorphic subsets of $\mathbb{C}^n$
    - Regular hypersurfaces in $\mathbb{R}^{2n}$
    - Orientable hypersurfaces in $\mathbb{R}^n$
    - Laplace operator on $\mathbb{R}^n$
    - $l^\infty(\mathbb{R})$
    - Lebesgue measurable subsets of and functions on $\mathbb{R}^n, T^n, S^n$
    - Lebesgue measure of boundary of open sets in $\mathbb{R}^n$
    - Local $\mathbb{R}$-algebras
    - $l^p(\mathbb{R})$
    - Metric density of subsets of $\mathbb{R}^n$
    - Monotone functions on $\mathbb{R}$
    - Cauchy integral of periodic functions
    - Polygons with integer vertices
    - Open smooth maps $U \subseteq \mathbb{R}^2 \to \mathbb{C}$
    - Discrete subgroups of $\mathbb{R}^n$
    - Discrete cocompact subgroups of $\mathbb{R}^n$, flat tori
    - Ramified germs of smooth and holomorphic functions
    - Open subsets of $\mathbb{R}^n$
    - Open subsets of $\mathbb{R}^n$ equipped with the flat metric
    - Closed subsets of $\mathbb{R}^n$ equipped with smooth functions
    - Quasi-analytic smooth functions on $\mathbb{R}$
    - Star-shaped subsets of $\mathbb{R}^n$
    - ODEs in $\mathbb{R}^n \leftrightarrow$ Vector fields in $\mathbb{R}^n$