Info

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Horospherical decomposition of $\mathsf{Pd}(n, \mathbb{R})$

Definition. Definition

- the **parabolic subgroup**

$$P_\xi := \{g \in GL(n,\mathbb{R}) | \boldsymbol{t}_g(\xi) = \xi\}$$

- the **horospherical subgroup**

$$N_\xi := \left\{ g \in G_\xi \middle| \exp(-tX) g \exp(t_X) \xrightarrow{t \to \infty} I \right\}$$

- $$F(c) := \bigcup \{\gamma(\mathbb{R}) | \gamma : \mathbb{R} \to \mathsf{Pd}(n,\mathbb{R}) \text{ is a geodesic parallel to } \xi\}$$

Proposition: Let $\xi \in \partial \mathsf{Pd}(n,\mathbb{R})$ and $X \in \mathsf{Sym}(n,\mathbb{R}), \|X\| = 1$ such that

$$\exp(tX) \xrightarrow{t \to \infty} \xi$$

. Then

$$P_\xi := \left\{ g \in GL(n,\mathbb{R}) \middle| \lim_{t \to \infty} \exp(-X) g \exp(tX) \in GL(n,\mathbb{R}) \right\}$$

Moreover,

$$OG_\xi O^{-1} = \left\{ \begin{bmatrix} A_{11} & A_{12} & & \\ & A_{22} & & \\ & & \ddots & \\ & & & A_{kk} \end{bmatrix} \in GL(n,\mathbb{R}) \middle| A_{ij} \in \mathsf{Mat}(r_i, r_j, \mathbb{R}) \right\}$$

Let

$$X = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

Then

$$[\exp(-tX) g \exp(-tX)]_{ij} = \exp(-t(\lambda_i - \lambda_j))[g]_{ij}$$

- $$\begin{aligned} &\left\{ d(\boldsymbol{t}_{\exp(-tX) g \exp(-tX)}(I), I) \middle| t \in \mathbb{R} \right\} \text{ is bounded} \\ &\implies d(\boldsymbol{t}_g(\exp(tX)), I) \end{aligned}$$
- ...

Proposition:

Proposition:

$$P_\xi \curvearrowright \mathsf{Pd}(n,\mathbb{R})$$

is transitive.

From Cholesky decomposition - Wikipedia we have $g = UU^\mathsf{T}$.

- ...

Proposition: Let $\exp(tX)$ be a geodesic ray in $\mathsf{Pd}(,\mathbb{R})$. Then

$$F(c) \subseteq \mathsf{Pd}(n,\mathbb{R})$$

consists of matrices in $\mathsf{Pd}(n,\mathbb{R})$ that commute with $\exp(X)$.

Given $p \in \mathsf{Pd}(n,\mathbb{R})$, $g \in G_\xi$ such that

$$p = \boldsymbol{t}_g(I) = gg^\mathsf{T}$$

Let c be a geodesic line such that $c(\infty) = \xi$. Then define $-\xi := c(-\infty)$.

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Proposition:

$$N_\xi \times F(c) \to \mathsf{Pd}(n,\mathbb{R})$$

is a bijection.

$$X = \begin{bmatrix} \mu_1 I_{r_1} & \\ & \mu_k I_{r_k} \end{bmatrix}$$

- **Base case:** for $k = 1$, N_ξ is trivial and $F(c) = \mathsf{Pd}(n,\mathbb{R})$
- **Induction hypothesis:** $vfv^\mathsf{T} = p$ has a unique solution in for $p \in \mathsf{Pd}(n,\mathbb{R})$ for $k-1$

- **Inductive step:** We have

$$
\begin{aligned}
&vfv^\mathsf{T} = p \\
\implies &\begin{bmatrix} v_{11} & v_{12} \\ & I_{r_k} \end{bmatrix} \begin{bmatrix} f_{11} & \\ & f_{22} \end{bmatrix} \begin{bmatrix} v_{11}^\mathsf{T} & \\ v_{12}^\mathsf{T} & I_k \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}
\end{aligned}
$$

  - > **Proposition:** $p_{11} - p_{12}p_{22}^{-1}p_{21} > 0$

  - $$\begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1(p_{11}y_1 + p_{21}y_2) + x_2(p_{12}y_1 + p_{22}y_2)]$$

Current note has 0 direct children and 0 total descendants.

- squishy
  - Pd
    - Pd$(n, \mathbb{R})$
      - Horospherical decomposition of Pd$(n, \mathbb{R})$

And it has 2 siblings.

- squishy
  - Pd
    - Pd$(n, \mathbb{R})$
      - $GL(n, \mathbb{R}) \curvearrowright$ Pd$(n, \mathbb{R})$
      - Horospherical decomposition of Pd$(n, \mathbb{R})$