Heat flow of a cocompact Fuchsian group

11.5 Weyl's Asymptotic Law

In the proof of the trace formula, we have used the "heat kernel" $h_t(r) = e^{-(\frac{1}{4}+r^2)t}$. The reason for this being called so is the following. The *Laplace operator* for hyperbolic geometry on $\mathbb{H}$,

$$\Delta = -y^2\left(\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2\right),$$

is invariant under G, i.e., $\Delta L_g = L_g\Delta$ for every $g \in G$. Therefore Δ defines a differential operator on the quotient $\Gamma\backslash\mathbb{H}$, which we denote by the same letter. It can be shown that its eigenvalues are $\lambda_j = (\frac{1}{4} + r_j^2)$ for $j \geq 0$. Since this requires additional arguments from Lie theory and is not essential for our purposes, we will not give the proof, but only mention the fact as an explanation for the terminology. The interested reader may consult Helgason's book [Hel01].

The hyperbolic *heat operator* is $e^{-t\Delta}$ for $t > 0$. This is an integral operator whose kernel $k_t(z, w)$ describes the amount of heat flowing in time t from point z to point w. Therefore

$$\sum_{j=0}^{\infty} e^{-\left(\frac{1}{4}+r_j^2\right)t} = \sum_{j=0}^{\infty} e^{-t\lambda_j} = \operatorname{tr} e^{-t\Delta}$$

is the *heat trace* on $\Gamma\backslash\mathbb{H}$.

Proposition 11.5.1 *As $t \to 0$, one has*

$$t\sum_{j=0}^{\infty} e^{-\left(\frac{1}{4}+r_j^2\right)t} \to \frac{\operatorname{vol}(\Gamma\backslash\mathbb{H})}{4\pi}.$$

11.6 The Selberg Zeta Function

As in the previous sections, let Γ be a torsion free hyperbolic uniform lattice in $\mathrm{SL}(2,\mathbb{R})$. The compact surface $\Gamma\backslash\mathbb{H}$ is homeomorphic to a 2-sphere with a finite number of handles attached. The number of handles g is ≥ 2. It is called the *genus* of the surface $\Gamma\backslash\mathbb{H}$ (See [Bea95]).

The Selberg zeta function for Γ is defined for $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$ as

$$Z(s) = \prod_{\gamma}\prod_{k\geq 0}\left(1 - e^{-(s+k)l(\gamma)}\right),$$

where the first product runs over all primitive hyperbolic conjugacy classes in Γ.

Theorem 11.6.1 *The product $Z(s)$ converges for $\operatorname{Re}(s) > 1$ and the function $Z(s)$ extends to an entire function with the following zeros. For $k \in \mathbb{N}$ the number $s = -k$ is a zero of multiplicity $2(g-1)(2k+1)$, where g is the genus of $\Gamma\backslash\mathbb{H}$. For every $j \geq 0$ the numbers*

$$\frac{1}{2} + ir_j, \quad \text{and} \quad \frac{1}{2} - ir_j$$

are zeros of $Z(s)$ of multiplicity equal to the multiplicity $N_\Gamma(\pi_{ir_j})$. These are all zeros.

Current note has 0 direct children and 0 total descendants.

- stem
  - Objects stemming from $PSL(2)(\mathbb{R})$
    - Fuchsian groups $\Gamma <_d PSL(2)(\mathbb{R})$ $\leftrightarrow$ oriented hyperbolic 2-manifold/orbifolds $\to$ Riemann surfaces with branching signature
      - Cocompact Fuchsian groups
        - Heat flow of a cocompact Fuchsian group

And it has 1 siblings.

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