Info

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Convex compact subsets of weak* dual

Let X be a $\mathbb{R}$-Banach space and $K \subseteq X^\star$ be a convex, weak*-compact subset of the dual space. Then K has an extreme point.

for separable Banach spaces

Let $\{x_n\}$ be a countable dense subset of X. Then

$$\mathrm{ev}_{x_n} : X^\star \to \mathbb{R}$$

- **Definition.**
$$l_1 := \sup_{f \in K} \underbrace{\mathrm{ev}_{x_1}(f)}_{f(x_1)}$$
- Since K is compact and ev_{x_1} is continuous, $l_1 < \infty$ is attained, say at $f_1 \in K$, that is
$$\mathrm{ev}_{x_1}(f_1) = l_1$$
- **Definition.**
$$l_2 := \sup_{f \in K, \mathrm{ev}_{x_1}(f) = l_1} \mathrm{ev}_{x_2}(f)$$
- Again, the set $K \cap \mathrm{ev}_{x_1}^{-1}(l_1)$ is compact, so $\exists f_2 \in K$ such that
$$\begin{aligned} \mathrm{ev}_{x_2}(f_2) &= l_2 \\ \text{and } \mathrm{ev}_{x_1}(f_2) &= l_1 \end{aligned}$$
- Inductively, we have a sequence
$$f_1, f_2, \dots \in K$$
such that for each n
$$\begin{aligned} \mathrm{ev}_{x_1}(f_n) &= l_1 \\ \mathrm{ev}_{x_2}(f_n) &= l_2 \\ &\dots \\ \mathrm{ev}_{x_n}(f_n) &= l_n \end{aligned}$$
and
$$l_{n+1} = \sup_{\substack{x \in K, \begin{subarray}{c} \mathrm{ev}_{x_1}(f) = l_1 \\ \dots \\ \mathrm{ev}_{x_n}(f) = l_n \end{subarray}}} \mathrm{ev}_{x+1}(f)$$
- Let f be an accumulation point of $\{x_n\}$. Then $f \in K$.
- For a fixed j,
$$\mathrm{ev}_{x_j}(f_n) = l_j$$
for all large n.
- Therefore
$$\mathrm{ev}_{x_j}(f) = l_j$$
for all j.

> **Proposition:** $f \in K$ is an extreme point

Let
$$f = \frac{1}{2}(y_1 + y_2)$$
for some $y_1, y_2 \in K$.

- Then evaluating x_1 we have
$$\begin{aligned} l_1 = \mathrm{ev}_{x_1}(f) &= \mathrm{ev}_{x_1}\left(\frac{1}{2}(y_1 + y_2)\right) \\ &= \frac{1}{2}(\mathrm{ev}_{x_1}(y_1) + \mathrm{ev}_{x_1}(y_2)) \end{aligned}$$
  - Since l_1 is the supremum of ev_{x_1} on K we have
$$\begin{aligned} \mathrm{ev}_{x_1}(y_1) &\le l_1 \\ \mathrm{ev}_{x_1}(y_2) &\le l_1 \end{aligned}$$
  - By
    Let F be an ordered field and $a, b \in F$. We have
$$\frac{a+b}{2} \in [a, b]$$

and

$$\begin{aligned} & \frac{a+b}{2} \in \{a, b\} \\ \Longleftrightarrow\ & a = b \\ \Longleftrightarrow\ & a = \frac{a+b}{2} = b \\ \Longleftrightarrow\ & \frac{a+b}{2} \leq a, \frac{a+b}{2} \leq b \end{aligned}$$

we have

$$\mathrm{ev}_{x_1}(y_1) = l_1 = \mathrm{ev}_{x_1}(y_2)$$

- Similarly,

$$\mathrm{ev}_{x_k}(y_1) = l_k = \mathrm{ev}_{x_k}(y_2)$$

- However, $\{x_k\}$ is dense in X. This implies

$$y_1 = y_2$$

- Therefore, $f \in K$ is extreme.

[1]

Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - Topological spaces
    - Topological vector spaces over $\mathbb{R}$ or $\mathbb{C}$
      - Convex compact subsets of weak* dual

And it has 7 siblings.

- structures based on sets
  - Topological spaces
    - Topological vector spaces over $\mathbb{R}$ or $\mathbb{C}$
      - Banach spaces
      - Dual space of a topological vector space
      - Balls in dual of a topological vector space
      - Convex compact subsets of weak* dual
      - Weak* topology on dual of a topological vector space
      - Compact endomorphisms
      - Weak topology on a topological vector space

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1. Herbert Alexander, John Wermer - Several Complex Variables and Banach Algebras (1998, Springer), p.17 ↩