Info

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Quasi-linear PDEs on $\mathbb{R}^2$

Definition. Definition

Let $O \subseteq \mathbb{R}^3$ be open and $a, b, c \in \mathcal{C}^1(O)$. For $U(\text{open}) \subseteq \pi_{1,2}(O)$ consider

$$Q(a,b,c)(U) := \left\{ u \in \mathcal{C}^1(U) \middle| \begin{array}{l} \forall (x,y) \in U \text{ we have } (x,y,u(x,y)) \in O \\ a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \end{array} \right\}$$

Bug

Show that this assignment $U \mapsto Q(a,b,c)$ is a sheaf on $\pi_{1,2}(O)$? Donno if that is the case.

- For any $u \in \mathcal{C}^1(U)$, the surface $S(u) := \{z = u(x,y)\}$ has normal $\begin{bmatrix} u_x \\ u_y \\ -1 \end{bmatrix}$.
  - Then
$$\begin{aligned} u \in Q(a,b,c)(U) &\iff \begin{bmatrix} u_x \\ u_y \\ -1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \\ &\iff (a,b,c)_{(x,y,z)} \in T_{(x,y,z)}S(u) \end{aligned}$$

given an initial curve

- Let Γ_0 be a $\mathcal{C}^1$-curve $\gamma : (0,1) \to U$ then
$$\begin{aligned} \text{res} : Q(a,b,c)(U) &\to \mathcal{C}^1(0,1) \\ u &\mapsto u \circ \gamma \end{aligned}$$
  be the restriction.
- Now consider $u_0 \in \mathcal{C}^1(0,1)$ such that
$$(\gamma, u_0) : (0,1) \to O$$
  be the *initial curve*.
  - We solve for $(X,Y,Z)(s,t)$ in
$$\begin{aligned} \begin{bmatrix} X_t \\ Y_t \\ Z_t \end{bmatrix} &= \begin{bmatrix} a \\ b \\ c \end{bmatrix}_{(X,Y,Z)} \\ (X,Y,Z)_{(s,0)} &= (\gamma, u_0)(s) \end{aligned}$$
    to obtain a $\mathcal{C}^1$
$$(X,Y,Z) : (0,1) \times (-\epsilon, \epsilon) \to O$$
    which maps
$$(s,0) \mapsto (\gamma, u_0)(s)$$
  - Let $s_0 \in (0,1)$ and consider
$$(X,Y) : (0,1) \times (-\epsilon, \epsilon) \to \pi_{1,2}(O)$$
    whose Jacobian is
$$\begin{bmatrix} X_s & X_t \\ Y_s & Y_t \end{bmatrix}$$
  - So, if we assume
$$\det \begin{bmatrix} \gamma'(s_0) & \begin{matrix} a(\gamma(s_0), u_0(s_0)) \\ b(\gamma(s_0), u_0(s_0)) \end{matrix} \end{bmatrix} \neq 0$$
    then there is a onb V of $(X,Y)_{(s_0,0)}$ inverse
$$(S,T) : V \subseteq \pi_{1,2}(O) \to (0,1) \times (-\epsilon, \epsilon)$$
  - This means
$$(x,y) \mapsto Z(S(x,y), T(x,y))$$
    is in $Q(a,b,c)(V)$.
  - This V depends on (γ, u).

Let $p_0 \in \pi_{1,2}(O)$. Then we can look at all solutions in arbitrary nb of p:

$$Q(a,b,c)_p := \frac{\bigsqcup_{p \in U \subseteq \pi_{1,2}(O)} Q(a,b,c)(U)}{u \sim v \iff \exists V_p : u|_{V_p} = v|_{V_p}}$$

Let $\gamma : (0,1) \to \pi_{1,2}(O)$ be a curve such that $\gamma(s_0) = p$. Then we may restrict

$$\begin{aligned} Q(a,b,c)_{p_0} &\to \mathcal{C}^1(0,1)_{s_0} \\ [u] &\mapsto [u \circ \gamma] \end{aligned}$$

and reconstruct

$$\begin{aligned} \left\{ u_0 \in \mathcal{C}^1(0,1)_{s_0} \middle| \det \begin{bmatrix} \gamma'(s_0) & \begin{matrix} a(\gamma(s_0), u_0(s_0)) \\ b(\gamma(s_0), u_0(s_0)) \end{matrix} \end{bmatrix} \neq 0 \right\} &\to Q(a,b,c)_{p_0} \\ u_0 &\mapsto u(x,y) := Z(S(x,y), T(x,y)) \end{aligned}$$

Current note has 0 direct children and 0 total descendants.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, _\Gamma\backslash^{\mathbb{C}^n}$
    - equations
      - Quasi-linear PDEs on $\mathbb{R}^2$

And it has 5 siblings.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, _\Gamma\backslash^{\mathbb{C}^n}$
    - equations
      - Equations for functions on $\mathbb{R}^2$
      - Quasi-linear PDEs on $\mathbb{R}^2$
      - $$\partial_t^2 + \sum_{i=1}^n v_i^2 \partial_{x_i}^2 + m^2$$
      - First order PDEs in $\mathbb{R}^n$
      - $$\partial_t^2 + \sum_{i=1}^n v_i^2 \partial_{x_i}^2$$