$(w^2-1)^2$

As an example, consider the polynomial equation

(5.1) $$w^4 - 2w^2 + 1 - z = 0.$$

A special solution is given by $(z,w) = (1,\sqrt{2})$ and near $z = 1$ this gives rise to the function $w_1(z) = \sqrt{1+\sqrt{z}}$ which solves (5.1); the convention here is that $\sqrt{z} > 0$ if $z > 0$. However, there are other solutions, namely $(z,w) = (1,-\sqrt{2})$ leading to $w_2(z) = -\sqrt{1+\sqrt{z}}$ as well as $(z,w) = (1,0)$. The latter formally corresponds to the functions $w(z) = \pm\sqrt{1-\sqrt{z}}$ which are not analytic near $z = 0$. Thus, $(1,0)$ is referred to as a branch point and it is characterized as a point obstruction to analytic continuation. The purpose of this chapter is to put this example, as well as all such algebraic equations, on a solid foundation in the context of Riemann surfaces.

For an explicit example of a Puiseux series, consider the irreducible polynomial

$$P(w,z) = w^4 - 2w^2 + 1 - z$$

and solve for w locally around $(w,z) = (1,0)$. This leads to

$$w(z) = \sqrt{1+\sqrt{z}} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} z^{\frac{n}{2}},$$

which converges in $|z| < 1$.

To conclude this section, let us now return full circle to the beginning of this chapter and construct the ramified Riemann surface defined by the germ $[\sqrt{1+\sqrt{z}}, 1]$. The convention regarding the square root shall be $\sqrt{x} > 0$ for $x > 0$.

This not only serves to illustrate the concepts which we have introduced here, but should also hopefully help in building some intuition for the more mechanical aspects of analytic continuation, algebraic germs, and the surfaces which they generate.

The aforementioned germ gives rise to the unique irreducible monomial $P(w,z) = (w^2-1)^2 - z$. The system $P = 0, \partial_w P = 0$ has solutions

$$(w,z) \in \{(0,1),(1,0),(-1,0)\}.$$

Note that $\partial_w^2 P(w,z) \neq 0$ as well as $\partial_z P(w,z) \neq 0$ at each of these points.

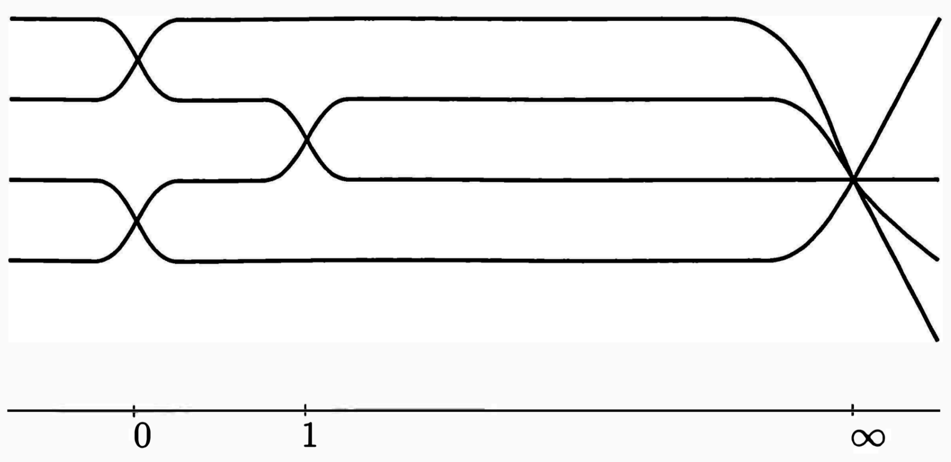

Figure 5.12. The sheets of $\sqrt{1+\sqrt{z}}$

Thus, any branch point associated with them has branching number 1. Moreover, if $z=0$, then necessarily $w=\pm 1$, whereas $z=1$ yields $w=0$ as well as $w=\pm\sqrt{2}$. Finally, $z=\infty$ is a branch point and analytic continuation around a large circle (more precisely, any loop encircling both $z=0$ and $z=1$ once) permutes the four sheets cyclically.

For this reason, the sheets look schematically as shown in Figure 5.12 (the two sheets which branch over $z=1$ cannot branch again over $z=0$ since this would contradict the aforementioned cyclic permutation property at $z=\infty$). Let us now analyze the four sheets and their permutation properties more carefully.

This is most easily done by introducing cuts from the branch points to infinity; cf. Figure 5.13. First, on the simply-connected region

$$\Omega := \mathbb{C} \setminus (-\infty, 0]$$

there exist four branches $f_j(z)$, $0 \le j \le 3$ uniquely determined by the asymptotic equalities

$$f_0(x) \sim x^{\frac{1}{4}},\ f_1(x) \sim ix^{\frac{1}{4}},\ f_2(x) \sim -x^{\frac{1}{4}},\ f_3(x) \sim -ix^{\frac{1}{4}}$$

as $x \to \infty$. The enumeration here has been chosen so that analytic continuation along any circle containing $z=0$ and $z=1$ with positive orientation induces the cyclic permutation

$$f_0 \mapsto f_1 \mapsto f_2 \mapsto f_3 \mapsto f_0.$$

Next, for all $x>1$ there are the explicit expressions

$$f_0(x) = \sqrt{1+\sqrt{x}}, \quad f_2(x) = -\sqrt{1+\sqrt{x}},$$
$$f_1(x) = i\sqrt{\sqrt{x}-1}, \quad f_3(x) = -i\sqrt{\sqrt{x}-1}.$$

Analytic continuation to Ω yields

$$f_1(i0+) = f_3(i0-) = f_2(0) = -1, \quad f_1(i0-) = f_3(i0+) = f_0(0) = 1.$$

Recall that by our convention, $\sqrt{z}$ is analytic on Ω with $\sqrt{x}>0$ if $x>0$. Analytic continuation around the loop $z(\theta) = 1+\varepsilon^2 e^{i\theta}$, $0 \le \theta \le 2\pi$, with $\varepsilon > 0$ small leaves f_0 and f_2 invariant, whereas f_1 and f_3 are interchanged. Similarly, analytic continuation around $z(\theta) = \varepsilon^2 e^{i\theta}$ (starting with $\theta > 0$ small) takes f_0 into f_3 and f_2 into f_1. This implies that the *monodromy group* of this Riemann surface is generated by the permutations (12)(03) and (13).

The formal definition of the monodromy group is as follows: for any $c \in \pi_1(\mathbb{C}\setminus\{0,1\})$ let $\mu(c) \in S_4$ (the group of permutations on four symbols) be defined as the permutation of the four sheets which is induced by analytic continuation along the closed loop c. For this we assume that c has its base point somewhere in Ω and we pick the four germs defined on each branch over that base point for analytic continuation. Note that μ is well-defined by the monodromy theorem.

This refers to the fact that c is really a representative of an equivalence class of loops which are all homotopic in $\mathbb{C}\setminus\{0,1\}$. The map

$$\mu : \pi_1(\mathbb{C}\setminus\{0,1\}) \to S_4$$

is a group homomorphism and the monodromy group is the image $\mu(\pi_1)$. For example, $(13)(12)(03) = (0123)$ which is precisely the cyclic permutation at $z=\infty$. The genus g of this Riemann surface is given by the Riemann-Hurwitz formula as

$$g-1 = 4(-1) + \frac{1}{2}(1+1+1+3) = -4+3 = -1$$

which implies that $g=0$.

Current note has 0 direct children and 0 total descendants.

- squishy
  - Polynomials
    - $(w^2-1)^2$

And it has 17 siblings.

- squishy
  - Polynomials
    - $X^n + Y^n = Z^n$
    - $8X^3 + 4X^2 - 4X - 1$
    - $x^2 + y^2 - 3$
    - Folium of Descartes

      $X^3 + Y^3 - 3aXY$

- $X^3 + Y^3 - a$
- $3X^3 + 10X^2 + 3 - Y^2$
- $3X^8 + 10X^4 + 3 - Y^2$
- $XYZ - 1$
- Pythagorean triples
- Bolza curve
- Irreducible polynomials in $\mathbb{C}[x, y]$
- Polynomial maps $\mathbb{C}^n \to \mathbb{C}^n$ of degree at most d
- Chebyshev polynomials of the first kind
- Logistic map
- Dynamics of the map $z^2 + c$
- $\begin{pmatrix} x \\ y \\ y \end{pmatrix} \mapsto \begin{pmatrix} (1+xy)^3 z + y^2(1+xy)(4+3xy) \\ y + 3x(1+xy)^2 z + 3xy^2(4+3xy) \\ 2x - 3x^2 y - x^3 z \end{pmatrix}$
- $(w^2 - 1)^2$