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Vector field on smooth manifold M as derivation of $\mathcal{C}^\infty(M)$

Derivations on $\mathcal{C}^\infty(M)$ can be identified with vector fields.

Definition. Action of vector fields on $\mathcal{C}^\infty(M)$

If $X \in \Gamma(TM)$, $f \in \mathcal{C}^\infty(M)$ define the multiplication

$$\Gamma(TM) \times \mathcal{C}^\infty(M) \rightarrow \Gamma(TM)$$

such that $(X, f) \mapsto Xf$ where

$$(Xf)(p) := X_p f$$

Let X be a rough vector field on M . Then

$$\begin{aligned} X &\in \Gamma(TM) \\ &\iff \\ f \in \mathcal{C}^\infty(M) &\implies Xf \in \mathcal{C}^\infty(M) \\ &\iff \end{aligned}$$

for any open subset $U \subseteq M$

$$f \in \mathcal{C}^\infty(U) \implies Xf \in \mathcal{C}^\infty(U)$$

So every smooth vector field $X \in \Gamma(TM)$ defines a map $X : \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M)$ by

$$f \mapsto Xf$$

which clearly

- is $\mathbb{R}$ -linear
- satisfies

$$f Xg + g Xf$$

Hence, $X : \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M)$ is a *derivation* on $\mathcal{C}^\infty(M)$. So every smooth vector field produces a derivation on $\mathcal{C}^\infty(M)$. The converse is also true.

A map $D : \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M)$ is a derivation

$$\iff$$

$$Df = Xf \text{ for some } X \in \Gamma(TM).$$

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