Info

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Functions $(a, b) \to \mathbb{R}$ differentiable at a point

Definition. Derivative of functions on $\mathbb{R}$

Let $f : I(\text{interval}) \subseteq \mathbb{R} \to \mathbb{R}$ be a function and $x \in I$. Then

$$\begin{aligned} I \setminus \{x\} &\to \mathbb{R} \\ t &\mapsto \frac{f(t) - f(x)}{t - x} \end{aligned}$$

is well-defined. Then $f'(x) \in \mathbb{R} \cup \{\infty, -\infty\}$ is defined to be

$$f'(x) := \lim_{(a,b)\setminus\{x\} \ni t \to x} \frac{f(t) - f(x)}{t - x}$$

whenever the right side exists in $\mathbb{R} \cup \{\infty, -\infty\}$.
The function f is called **differentiable** at p if $f'(p) \in \mathbb{R}$.
Higher derivatives are defined iteratively

$$\begin{aligned} f^{(0)} &:= f \\ f^{(n)}(x) &:= (f^{(n-1)})'(x) \end{aligned}$$

whenever they exist.

- By definition, derivative at boundary points ∂I makes sense.
- The function

$$\begin{aligned} (a, b) \times [a, b] \setminus \{t = x\} &\to \mathbb{R} \\ (t, x) &\mapsto \frac{f(t) - f(x)}{t - x} \end{aligned}$$

derivative

- Let $f : (a, b) \to \mathbb{R}$ be differentiable at $c \in (a, b)$.
  - Define

$$f_c^*(x) := \begin{cases} \dfrac{f(x) - f(c)}{x - c} & \text{if } x \neq c \\ f'(c) & \text{if } x = c \end{cases}$$

      - f^* is continuous in (a, b).
  - So we can write f as

$$f(x) - f(c) = (x - c) f_c^*(x)$$

    for an unique continuous f_c^*.
- Assume can write a function $f : (a, b) \to \mathbb{R}$ as

$$f(x) - f(c) = (x - c) f_c^*(x)$$

  for some continuous f_c^* for some $c \in (a, b)$.
  - Then for any $x \neq c$, taking

$$\frac{f(x) - f(c)}{x - c} = f_c^*(x)$$

    and then the limit $x \to c$, shows $f'(c) = f_c^*(c)$.

Thus, we proved that

A function $f : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b) \iff \exists$ a continuous function $f_c^* : (a, b) \to \mathbb{R}$ such that

$$f(x) = f(c) + (x - c) f_c^*(x)$$

for all $x \in (a, b)$. In any case $f'(c) = f_c^*(c)$.

- From
  > - Let $f : (a, b) \to \mathbb{R}$ be differentiable at $c \in (a, b)$.
  >   - Define
  >
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  >
  >   - f^* is continuous in (a, b).
  >   - So we can write f as
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  > $$f(x) - f(c) = (x - c) f_c^*(x)$$
  >
  >   for an unique continuous f_c^*.

  we write

$$f(x) = f(c) + (x - c) f_c^*(x)$$

  as

$$f(x) = f(x) + (x - c) f'(c) + (x - c) h_c(x)$$

  where

$$\begin{aligned} h_c(x) &:= f_c^*(x) - f(x) \\ \implies \lim_{x \to c} h_c(x) &= 0 \end{aligned}$$

$f:(a,b)\to\mathbb{R}$ being differentiable at $c\in(a,b) \implies f$ is continuous at c.

non-zero derivative

(Strictly positive derivative $\implies$ locally (strictly) increasing) Let $f:(a,b)\to\mathbb{R}$ be differentiable at $c\in(a,b)$ and

$$f'(c)>0 \text{ or } +\infty$$

then there is a $I(c)\subseteq(a,b)$ in which $f(x)-f(c)$ has same sign as $x-c$, that is, f is strictly increasing on $I(c)$.

derivative at extremums

(Derivative, if it exists, must be 0 at local extremas) Let a function $f:(a,b)\to\mathbb{R}$ have a local maxima or minima at $c\in(a,b)$. If f has an extended derivative at c then the derivative must be zero

$$f'(c)\in\mathbb{R}\cup\{-\infty,\infty\} \implies f'(c)=0$$

Therefore,

$$\{\text{local extremum of } f\}\subseteq Z(f')$$

Current note has 0 direct children and 0 total descendants.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, _\Gamma\backslash^{\mathbb{C}^n}$
    - Differentiable functions
      - Functions $(a,b)\to\mathbb{R}$ differentiable at a point

And it has 10 siblings.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, _\Gamma\backslash^{\mathbb{C}^n}$
    - Differentiable functions
      - Functions $(a,b)\to\mathbb{R}$ differentiable at a point
      - Comparing a function and it derivative
      - Absolutely continuous functions on $[a,b]$ $\leftrightarrow$ $\int_{[a,-]}(L^1[a,b])$
      - Distributional derivatives
      - Double derivative/Laplace operator on the circle
      - Fractional derivative
      - Infinite limit of derivatives
      - Space of continuous and continuously differentiable functions on $\mathbb{R}$
      - Derivative of maps $\mathbb{R}^n\to\mathbb{R}^m$
      - Zooming of a map $\mathbb{R}^n\to\mathbb{R}^m$