ⓘ Info

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Flows on a smooth manifold

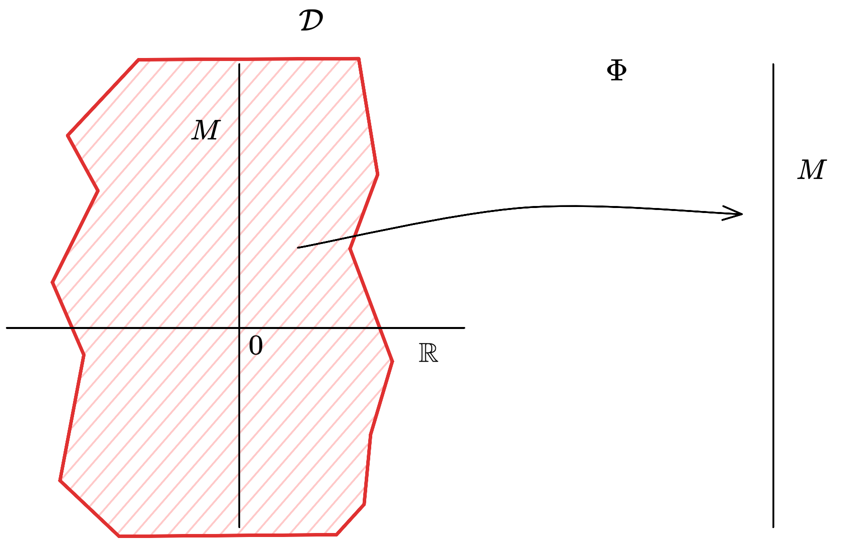
Definition. Flows on a smooth manifold

A flow on a smooth manifold M

$$\Phi^t$$

is a smooth map

$$\begin{aligned} \Phi : \mathcal{D}\,(\text{open}) \subseteq \mathbb{R} \times M &\to M \\ (t, p) &\mapsto \Phi^t(p) \end{aligned}$$



where

- $\{0\} \times M \subseteq \mathcal{D}$
- $\Phi_0 = \mathsf{Id}_M$
- $\Phi_{t_1} \circ \Phi_{t_2} = \Phi_{t_1+t_2}$

Definition. Generator of a flow on a smooth manifold is a vector field

The **generator of a flow** Φ^t on M is the vector field

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi^t(p)\Big|_{t=0} =: X_p^\Phi$$

For each t, if Φ^t is a local diffeomorphism. It is a "smooth" curve in $\mathrm{LDiff}(M)$.

Definition. Flow and exponential of a vector field on a smooth manifold

Flow of a vector field $X \in \mathrm{Vec}(M)$ is the *flow* on M, that is, the smooth map

$$(t, -) \mapsto (\exp tX)(-)$$

that satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp tX\Big|_{t=0}(p) = X_p, \quad \exp 0 = \mathsf{Id}_M$$

Then X is the **generator of the flow** and

$$\exp X : M \to M$$

is the **exponential of the vector field** X.

By existence and uniqueness we have a global correspondence

$$\text{vector fields on } M \quad \underset{\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}}{\overset{\exp(t-)}{\rightleftharpoons}} \quad \text{flows on } M$$

where do we find complete flows

- vector fields with compact support has complete flows
  - vector fields on compact manifolds have complete flows
- Smooth Lie group action on a smooth manifold produces complete flows

set of all flows

We must quotient two flows if they agree on a ϵ-interval near 0 in time:

$$\mathrm{Flow}(U) = \frac{\{\Phi : D \subseteq \mathbb{R} \times U \to U \mid \Phi \text{ is a flow on } U\}}{\Phi_1 \sim \Phi_2 \text{ if } \exists \epsilon, |t| \le \epsilon \implies \Phi_1^t = \Phi_2^t}$$

Flow and Vec sheaves

$$\mathrm{Flow} : \mathrm{Open}(M)^{op} \to \mathsf{Set}$$

is a pre-sheaf (functor).

Let's call its sheaffification

$$\mathrm{Flow}^{sh}$$

- map of sheaves

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} : \mathrm{Flow}^{sh} &\to \mathrm{Vec} \\ \mathrm{Flow}^{sh}(U) &\to \mathrm{Vec}(U) \\ \Phi^t &\mapsto \left.\frac{\mathrm{d}}{\mathrm{d}t}\Phi^t\right|_{t=0} \end{aligned}$$

- map of sheaves

$$\begin{aligned} \exp(t-) : \mathrm{Vec} &\to \mathrm{Flow}^{sh} \\ \mathrm{Vec}(U) &\to \mathrm{Flow}^{sh}(U) \\ X &\mapsto \exp tX \end{aligned}$$

  - isomorphism of sheaves? [1]
  - on stalks, this give an isomorphism?
- Constants of the flow on a smooth manifold
-

all flows away from fixed points are locally parallel straight lines

Let X be a vector field on a smooth manifold M such that $X_p \neq 0$ at some $p \in M$. Then there exists a local chart near p such that trajectories of the local flow of X in this local chart are **parallel straight lines**.

on hyperbolic fixed points

Let X is a smooth vector field on M and p is a hyperbolic fixed point of its flow $\exp(tX)$. Then the flow locally near p is **topologically equivalent** to the flow $\exp(t\mathrm{d}_pX)$ on T_pM near 0.

Current note has 1 direct children and 1 total descendants.

- structures based on sets
  - manifolds
    - smooth $\mathbb{R}$-manifold
      - Flows on a smooth manifold
        - Constants of the flow on a smooth manifold

And it has 19 siblings.

- structures based on sets
  - manifolds
    - smooth $\mathbb{R}$-manifold
      - Smooth Lie group action on a smooth manifold
      - Borel measures on a smooth manifold
      - G-bundles over smooth manifolds $G \curvearrowright P \to X$
      - Vector bundle on smooth manifolds
      - Curves in smooth manifolds
      - Density on smooth manifolds
      - Functions between smooth manifolds
      - Functions from smooth manifolds to $\mathbb{R}$
      - Flows on a smooth manifold
      - Locally $G \curvearrowright X$-manifolds
      - Smooth singular homology of a smooth manifold
      - Isomorphisms and automorphisms of smooth manifolds
      - Lagrangian flows
      - Local transformation rules
      - Moduli space of Riemannian metrics on a smooth manifold
      - Moduli space of quotients
      - Quotient manifolds
      - Submanifolds of smooth manifolds
      - Tangents on a smooth manifold

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1. differential geometry - Flow of a Vector Field Using Sheaves - Mathematics Stack Exchange ↩