Info

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$GL(n,\mathbb{R}) \curvearrowright \mathsf{Pd}(n,\mathbb{R})$

Definition. The transvection action $GL(n,\mathbb{R}) \curvearrowright \mathsf{Pd}(n,\mathbb{R})$

The smooth action

$$\begin{aligned} \boldsymbol{t} : GL(n,\mathbb{R}) &\curvearrowright \mathsf{Pd}(n,\mathbb{R}) \\ p &\overset{g}{\mapsto} \boldsymbol{t}_g(p) := gpg^{\mathsf{T}} \end{aligned}$$

is called transvection.

> **Proposition:** The action
>
> **Definition.** The transvection action $GL(n,\mathbb{R}) \curvearrowright \mathsf{Pd}(n,\mathbb{R})$
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> The smooth action
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>
> is called transvection.
>
> is transitive with stabilizer of $I \in \mathsf{Pd}(n,\mathbb{R})$ being $O(n,\mathbb{R})$.

Diagonalize A using a $O \in SO(n,\mathbb{R})$ so

$$OAO^{\mathsf{T}} = D \implies A = O^{\mathsf{T}}AO$$

now

$$g := O^{\mathsf{T}}\sqrt{D}O$$

means

$$g = g^{\mathsf{T}}, A = gg^{\mathsf{T}}$$

Thus

$$A = gIg^{\mathsf{T}}$$

homogeneous metric

> **Proposition:**
>
> - The action of $GL(n,\mathbb{R})$ is by isometries
>
> $$GL(n,\mathbb{R}) \curvearrowright (\mathsf{Pd}(n,\mathbb{R}), g_{\mathsf{Pd}})$$
>
> - The complete metric space $(\mathsf{Pd}(n,\mathbb{R}), d_{\mathsf{Pd}})$ is CAT(0).
> - The exponential of matrices is the Riemannian exponential
>
> $$\exp = \exp_I$$
>
> - The inclusion
>
> $$GL(n,\mathbb{R}) \to \mathrm{Isom}(\mathsf{Pd}(n,\mathbb{R}), g_{\mathsf{Pd}})$$
>
> is open? and surjective? onto the connected component of isometry group with kernel
>
> $$\{I, -I\}$$
>
> - The map
>
> $$\begin{aligned} \sigma_I : \mathsf{Pd}(n,\mathbb{R}) &\to \mathsf{Pd}(n,\mathbb{R}) \\ q &\mapsto q^{-1} \end{aligned}$$
>
> is an isometry which fixes I and whose differential at I is $-\mathsf{Id}$. By homogeneity we get a geodesic symmetry at $p \in \mathsf{P}(n,\mathbb{R})$
>
> $$\begin{aligned} \sigma_p : \mathsf{Pd}(n,\mathbb{R}) &\to \mathsf{Pd}(n,\mathbb{R}) \\ q &\mapsto pq^{-1}p \end{aligned}$$
>
> Thus, $(\mathsf{Pd}(n,\mathbb{R}), g_{\mathsf{Pd}})$ **is a CAT(0) symmetric space**.

dimension counts

Fix $n > 1$. Then

$$\dim GL(n,\mathbb{R}) = n^2$$

$$\frac{O(n,\mathbb{R})}{\{I,-I\}} \hookrightarrow O(\mathsf{Sym}(n,\mathbb{R}))$$

whose difference in dimension is

$$\begin{aligned} \dim O(\mathsf{Sym}(n,\mathbb{R})) &= \frac{n(n+1)}{2}\left(\frac{n(n+1)}{2} - 1\right) \\ \dim O(n,\mathbb{R}) &= \frac{n(n-1)}{2} \\ \dim O(\mathsf{Sym}(n,\mathbb{R})) - \dim O(n,\mathbb{R}) &= \frac{n(n+1)}{2}\left(\frac{n(n+1)}{2} - 1\right) - \frac{n(n-1)}{2} \\ &= \frac{n^2(n^2+2n-3)}{4} \end{aligned}$$

Current note has 0 direct children and 0 total descendants.

- squishy
    - Pd
        - $\mathsf{Pd}(n, \mathbb{R})$
            - $GL(n, \mathbb{R}) \curvearrowright \mathsf{Pd}(n, \mathbb{R})$

And it has 2 siblings.

- squishy
    - Pd
        - $\mathsf{Pd}(n, \mathbb{R})$
            - $GL(n, \mathbb{R}) \curvearrowright \mathsf{Pd}(n, \mathbb{R})$
            - Horospherical decomposition of $\mathsf{Pd}(n, \mathbb{R})$