Info

This note, that was found here, as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
was created on March 30, 2026 6:21:05 PM,
was last modified on April 10, 2026 6:10:51 PM, and
was exported on September 7, 2026 3:09:47 PM.

Central simple algebra

Definition. Central simple algebra

Let F be a field. An associative algebra A over F is called **central simple algebra** over F if

$$Z(A) = F$$

and the only two-sided ideals in A are (0) and A.

Example

$$A = \mathrm{Mat}(n)(F)$$

quaternion algebra over F

Definition. Let $a, b \in F \setminus \{0\}$ then

$$\mathbb{H}^{a,b}(F) := F\left\langle i, j, ij \middle| i^2 = a, j^2 = b, ij = -ji \right\rangle$$

Proposition:

$$\mathbb{H}^{a,b}(F) \cong \mathbb{H}^{\alpha^2 a, \beta^2 b}(F)$$

Definition. Definition

$$\bar{q} := x - yi - zj - wk$$

then norm is

$$N(q) := q\bar{q}$$

$q \in \mathbb{H}^{a,b}(F)$ is a unit $\iff N(q) \neq 0$

Let $k = \bar{k}$. Then any finite dim division ring D over k such that $Z(D) = k$ is F itself.

$\mathbb{H}^{5,7}(\mathbb{Q})$ is a division ring

Let $(x_0, \dots, x_3) \in \mathbb{Z}^4$ primitive and

$$N(\alpha) = x_0^2 - 5x_1^2 - 7x_2^2 + 35x_3^2 = 0$$

Then

$$7 \mid x_0^2 - 5x_1^2$$

Fuchsian groups from quaternion algebras over totally real algebraic number fields

Let $F \subset \mathbb{R}$ be a totally real algebraic number field and $\sigma : F \to \mathbb{C}$ be an embedding. Then $A^\sigma \otimes \mathbb{R}$ is either

- $\mathrm{Mat}(2)(\mathbb{R})$
- $\mathbb{H}$
  - in this case A is a division ring

Let $\mathbb{Q} < F$ be a finite extension and $A := \mathbb{H}^{a,b}(F)$. Then a **order** $\mathcal{O}$ of A is a $\mathcal{O}_F$-subalgebra of A, finitely generated as a $\mathcal{O}_F$-module, such that

$$A = F[\mathcal{O}] = F\mathcal{O}$$

Then

$$\begin{array}{ccc} \mathcal{O} & \hookrightarrow & A \\ \uparrow & & \uparrow \\ \mathcal{O}_F & \hookrightarrow & F \end{array}$$

Let $\mathbb{Q} \leq F$ be a totally real algebraic number field and $A := \mathbb{H}^{a,b}(F)$. Let $\rho_1, \dots, \rho_n : F \hookrightarrow \mathbb{R}$ be the embeddings such that

$$\underbrace{A^{\rho_1} \otimes \mathbb{R}}_{\varphi_1 \cong \mathrm{Mat}(2)(\mathbb{R})}, \underbrace{\dots, A^{\rho_n} \otimes \mathbb{R}}_{\cong \mathbb{H}}$$

Then

$$\varphi_1(U(\mathcal{O})) \subseteq \mathrm{Mat}(2)(\mathbb{R})$$

Weil's restriction of scalars

Current note has 0 direct children and 0 total descendants.

- stem
  - Objects stemming from $PSL(2)(\mathbb{R})$

- Fuchsian groups $\Gamma <_d PSL(2)(\mathbb{R})$ $\leftrightarrow$ oriented hyperbolic 2-manifold/orbifolds $\to$ Riemann surfaces with branching signature
    - Central simple algebra

And it has 17 siblings.

- stem
  - Objects stemming from $PSL(2)(\mathbb{R})$
    - Fuchsian groups $\Gamma <_d PSL(2)(\mathbb{R})$ $\leftrightarrow$ oriented hyperbolic 2-manifold/orbifolds $\to$ Riemann surfaces with branching signature
      - Artihmetic Fuchsian groups and Artihmetic hyperbolic 2-orbifolds
      - Cocompact Fuchsian groups
      - Central simple algebra
      - Elementary Fuchsian groups
      - Finitely generated Fuchsian groups $\leftrightarrow$ geometrically finite Fuchsian groups
      - Fuchsian groups of finite coarea
      - Geodesic flow of Fuchsian groups and hyperbolic surfaces
      - Decomposition of hyperbolic surfaces
      - Infinitely generated Fuchsian groups
      - Limit set of Fuchsian groups
      - Marked hyperbolic surfaces
      - subst.R PSL 2 discrete subg.over Q-bar
      - Fuchsian Schottky groups
      - Spectrum of hyperbolic surfaces
      - Torsion-free Fuchsian groups
      - Cocompact torsion-free Fuchsian groups
      - Fuchsian triangle groups