

This note, that [was found here](#), as a part of [a collection](#) is written (completely with human hands) by [Rupadarshi Ray](#), was created on May 10, 2025 11:35:33 AM, was last modified on June 18, 2026 5:28:54 PM, and was exported on September 7, 2026 3:31:23 PM.

Differential forms on complex manifolds

We have the $\mathbb{R}$ -algebra of $\mathbb{R}$ -valued differential forms on M

$$\Omega^\bullet(M, \mathbb{R})$$

and its p -th $\mathbb{R}$ -de Rham cohomology

$$H^p(\Omega^\bullet(M, \mathbb{R}))$$

We also have

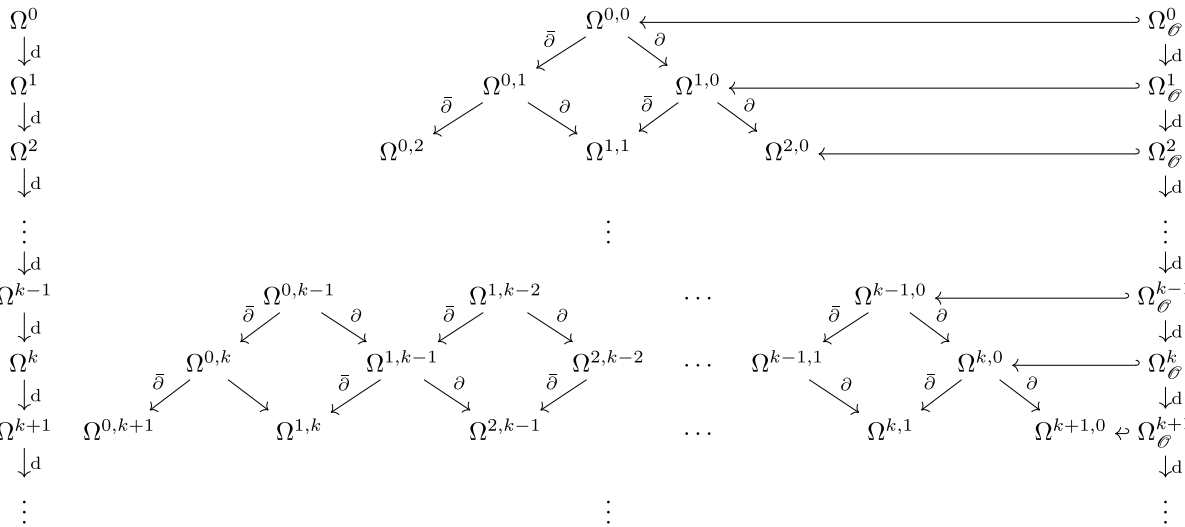
Proposition: The $\mathbb{C}$ -algebra of $\mathcal{C}^\infty(M, \mathbb{C})$ -differential forms on M

$$\Omega^\bullet(M, \mathbb{C})$$

has its p -th $\mathbb{C}$ -de Rham cohomology

$$H^p(\Omega^\bullet(M, \mathbb{C})) \cong H^p(\Omega^\bullet(M, \mathbb{R})) \otimes \mathbb{C}$$

- $\Omega^k(M, \mathbb{C}) \cong \bigoplus_{p+q=k} \Omega^{p,q}(M, \mathbb{C})$
- $\partial, \bar{\partial}$



Definition. Holomorphic forms, complex and cohomology

Let M be a complex manifold. **Holomorphic q -forms** on M are the elements of

$$\Omega^q_{\mathcal{O}}(M) := \Omega^{q,0}(M, \mathbb{C}) \cap \ker \bar{\partial}$$

which have elements of the form

$$\sum_{|I|=q} \phi_I(z) dz_I$$

where ϕ_I are holomorphic functions.

In particular, $\Omega^0_{\mathcal{O}} = \mathcal{O}$

We have the **Holomorphic de Rham complex**

$$\mathcal{O} \xrightarrow{d} \Omega^1_{\mathcal{O}} \xrightarrow{d} \Omega^2_{\mathcal{O}} \rightarrow \dots \xrightarrow{d} \Omega^{\dim(M)}_{\mathcal{O}}$$

and its cohomology

$$H^k(\Omega^\bullet_{\mathcal{O}}(M))$$

Definition. Dolbeault complex and cohomology

Fixing a q , we have

$$\Omega^{q,0} \xrightarrow{\bar{\partial}} \Omega^{q,1} \xrightarrow{\bar{\partial}} \Omega^{q,2} \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \Omega^{q,k} \rightarrow \Omega^{q,k+1} \rightarrow \dots$$

As as $\bar{\partial}^2 = 0$, this is a chain complex called **q -th Dolbeault chain complex**

$$(\Omega^{q,\bullet}, \bar{\partial})$$

The p -th cohomology of this chain complex is the **(p, q) -th Dolbeault cohomology**

$$H^p(\Omega^{q,\bullet}(M, \mathbb{C}), \bar{\partial})$$

(Dolbeault) Let M be a complex manifold. Then the q -th *sheaf cohomology* of sheaf of holomorphic p -forms is isomorphic to the (p, q) -th Dolbeault cohomology

$$H^q(M, \Omega^p_{\mathcal{O}}) \cong H^p(\Omega^{q,\bullet}(M, \mathbb{C}), \bar{\partial})$$

Current note has 0 direct children and 0 total descendants.

- [structures based on sets](#)
 - [manifolds](#)

- $\mathbb{C}$ -manifold
 - Tangent bundles on a complex manifold
 - Differential forms on complex manifolds

And it has 2 siblings.

- structures based on sets
 - manifolds
 - $\mathbb{C}$ -manifold
 - Tangent bundles on a complex manifold
 - Canonical almost complex structure on a complex manifold
 - Differential forms on complex manifolds