ⓘ Info

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is written (completely with human hands) by Rupadarshi Ray,
was created on May 10, 2025 11:35:33 AM,
was last modified on June 18, 2026 5:28:54 PM, and
was exported on September 7, 2026 3:31:23 PM.

Differential forms on complex manifolds

We have the $\mathbb{R}$-algebra of $\mathbb{R}$-valued differential forms on M

$$\Omega^\bullet(M,\mathbb{R})$$

and its p-th $\mathbb{R}$-*de Rham cohomology*

$$H^p(\Omega^\bullet(M,\mathbb{R}))$$

We also have

> **Proposition:** The $\mathbb{C}$-algebra of $\mathcal{C}^\infty(M,\mathbb{C})$-differential forms on M
>
> $$\Omega^\bullet(M,\mathbb{C})$$
>
> has its p-th $\mathbb{C}$-*de Rham cohomology*
>
> $$H^p(\Omega^\bullet(M,\mathbb{C})) \cong H^p(\Omega^\bullet(M,\mathbb{R}))\otimes\mathbb{C}$$

> - $$\Omega^k(M,\mathbb{C}) \cong \bigoplus_{p+q=k}\Omega^{p,q}(M,\mathbb{C})$$
> - $$\partial,\bar\partial$$

$$\begin{tikzcd}
\Omega^0 \arrow[d,"\mathrm{d}"] & & & & & & \Omega^{0,0} \arrow[dl,"\bar\partial"'] \arrow[dr,"\partial"] & & & & & & \Omega^0_{\mathscr{O}} \arrow[llllll, hook'] \arrow[d,"\mathrm{d}"] \\
\Omega^1 \arrow[d,"\mathrm{d}"] & & & & & \Omega^{0,1} \arrow[dl,"\bar\partial"'] \arrow[dr,"\partial"] & & \Omega^{1,0} \arrow[dl,"\bar\partial"'] \arrow[dr,"\partial"] & & & & & \Omega^1_{\mathscr{O}} \arrow[lllll, hook'] \arrow[d,"\mathrm{d}"] \\
\Omega^2 \arrow[d,"\mathrm{d}"] & & & & \Omega^{0,2} & & \Omega^{1,1} & & \Omega^{2,0} & & & & \Omega^2_{\mathscr{O}} \arrow[llll, hook'] \arrow[d,"\mathrm{d}"] \\
\vdots \arrow[d,"\mathrm{d}"] & & & & & & \vdots & & & & & & \vdots \arrow[d,"\mathrm{d}"] \\
\Omega^{k-1} \arrow[d,"\mathrm{d}"] & & & \Omega^{0,k-1} \arrow[dl,"\bar\partial"'] \arrow[dr,"\partial"] & & \Omega^{1,k-2} \arrow[dl,"\bar\partial"'] \arrow[dr,"\partial"] & & \cdots & & \Omega^{k-1,0} \arrow[dl,"\bar\partial"'] \arrow[dr,"\partial"] & & & \Omega^{k-1}_{\mathscr{O}} \arrow[lll, hook'] \arrow[d,"\mathrm{d}"] \\
\Omega^k \arrow[d,"\mathrm{d}"] & & \Omega^{0,k} \arrow[dl,"\bar\partial"'] \arrow[dr] & & \Omega^{1,k-1} \arrow[dl,"\bar\partial"'] \arrow[dr,"\partial"] & & \Omega^{2,k-2} \arrow[dl,"\bar\partial"'] & \cdots & \Omega^{k-1,1} \arrow[dr,"\partial"] & & \Omega^{k,0} \arrow[dl,"\bar\partial"'] \arrow[dr,"\partial"] & & \Omega^k_{\mathscr{O}} \arrow[ll, hook'] \arrow[d,"\mathrm{d}"] \\
\Omega^{k+1} \arrow[d,"\mathrm{d}"] & \Omega^{0,k+1} & & \Omega^{1,k} & & \Omega^{2,k-1} & & \cdots & & \Omega^{k,1} & & \Omega^{k+1,0} & \Omega^{k+1}_{\mathscr{O}} \arrow[l, hook'] \arrow[d,"\mathrm{d}"] \\
\vdots & & & & & & \vdots & & & & & & \vdots
\end{tikzcd}$$

Definition. Holomorphic forms, complex and cohomology

Let M be a complex manifold. **Holomorphic q-forms** on M are the elements of

$$\Omega^q_{\mathscr{O}}(M) := \Omega^{q,0}(M,\mathbb{C}) \cap \ker\bar\partial$$

which have elements of the form

$$\sum_{|I|=q}\phi_I(z)\mathrm{d}z_I$$

where ϕ_I are holomorphic functions.
In particular, $\Omega^0_{\mathscr{O}} = \mathscr{O}$
We have the **Holomorphic de Rham complex**

$$\mathscr{O} \xrightarrow{\mathrm{d}} \Omega^1_{\mathscr{O}} \xrightarrow{\mathrm{d}} \Omega^2_{\mathscr{O}} \to \dots \xrightarrow{\mathrm{d}} \Omega^{\dim(M)}_{\mathscr{O}}$$

and its cohomology

$$H^k(\Omega^\bullet_{\mathscr{O}}(M))$$

Definition. Dolbeault complex and cohomology

Fixing a q, we have

$$\Omega^{q,0} \xrightarrow{\bar\partial} \Omega^{p,1} \xrightarrow{\bar\partial} \Omega^{q,2} \xrightarrow{\bar\partial} \dots \xrightarrow{\bar\partial} \Omega^{q,k} \to \Omega^{q,k+1} \to \dots$$

As as $\bar\partial^2 = 0$, this is a chain complex called **q-th Dolbeault chain complex**

$$\left(\Omega^{q,\bullet},\bar\partial\right)$$

The p-th cohomology of this chain complex is the **(p,q)-th Dolbeault cohomology**

$$H^p\left(\Omega^{q,\bullet}(M,\mathbb{C}),\bar\partial\right)$$

(Dolbeault) Let M be a complex manifold. Then the q-th *sheaf cohomology* of sheaf of holomorphic p-forms is isomorphic to the (p,q)-th Dolbeault cohomology

$$H^q(M,\Omega^p_{\mathscr{O}}) \cong H^p\left(\Omega^{q,\bullet}(M,\mathbb{C}),\bar\partial\right)$$

Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - manifolds

- $\mathbb{C}$-manifold
  - Tangent bundles on a complex manifold
    - Differential forms on complex manifolds

And it has 2 siblings.

- structures based on sets
  - manifolds
    - $\mathbb{C}$-manifold
      - Tangent bundles on a complex manifold
        - Canonical almost complex structure on a complex manifold
        - Differential forms on complex manifolds