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Riemann-Darboux and Riemann-Stieltjes integrals

Let $[a, b]$ be a compact interval. A **partition** of $[a, b]$ is a finite set of points

$$\{x_k | 1 \leq k \leq n\} \subset [a, b]$$
$$a = x_0 \leq x_1 \leq \cdots \leq x_n = b$$

in $[a, b]$. The differences are

$$\Delta x_i := x_i - x_{i-1}$$

for each $1 \leq i \leq n$.

Definition. Darboux integral

Let

$$f : [a, b] \rightarrow \mathbb{R}$$

be bounded. Then the **upper Darboux sum** $\overline{\int}_{[a, b]} f$ and **lower Darboux sum** $\int_{-[a, b]} f$ of f are the following

$$\begin{aligned} & \left(\inf_{[a, b]} f \right) |b - a| \\ \leq \int_{-[a, b]} f &:= \sup \left\{ \sum_{1 \leq k \leq n} \left(\inf_{[x_{i-1}, x_i]} f \right) \Delta x_i \mid \begin{array}{l} \{x_0, \dots, x_n\} \subset [a, b] \\ \text{is a partition} \\ \text{for some } n \geq 1 \end{array} \right\} \\ \leq \overline{\int}_{[a, b]} f &:= \inf \left\{ \sum_{1 \leq k \leq n} \left(\sup_{[x_{i-1}, x_i]} f \right) \Delta x_i \mid \begin{array}{l} \{x_0, \dots, x_n\} \subset [a, b] \\ \text{is a partition} \\ \text{for some } n \geq 1 \end{array} \right\} \\ & \leq \left(\sup_{[a, b]} f \right) |b - a| \end{aligned}$$

If the upper and lower Darboux sum are equal, then the **Darboux integral** of f on $[a, b]$ is

$$\int_{[a, b]} f := \overline{\int}_{[a, b]} f = \int_{-[a, b]} f$$

Definition. Riemann integral

integration and differentiation

Let f be Riemann-Darboux integrable on $[a, b]$ and let F be differentiable on $[a, b]$ such that $F' \equiv f$. Then

$$\int_{[a, b]} f = F(b) - F(a)$$

for Riemann integral

for Darboux integral

Let $\epsilon > 0$. Then as f is Riemann-Darboux integrable there exists a partition $\{x_0, \dots, x_n\} \subset [a, b]$ so that

$$\left| \sum_{1 \leq k \leq n} \left(\inf_{[x_{i-1}, x_i]} f \right) \Delta x_i - \sum_{1 \leq k \leq n} \left(\sup_{[x_{i-1}, x_i]} f \right) \Delta x_i \right| < \epsilon$$

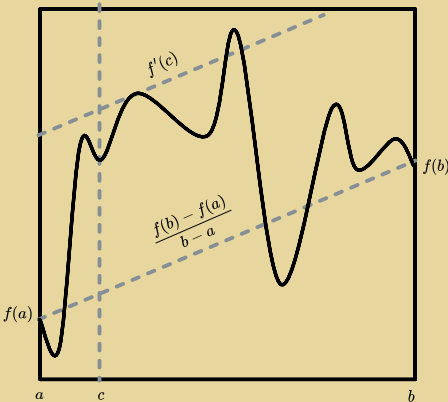
- Then by

(Lagrange's mean value theorem) Let a continuous function

$$f : [a, b] \rightarrow \mathbb{R}$$

be differentiable on (a, b) . Then there is a point $c \in (a, b)$ at which

$$f(b) - f(a) = (b - a)f'(c)$$



there are points $t_i \in [x_{i-1}, x_i]$ such that

$$F(x_i) - F(x_{i-1}) = f(t_i)\Delta x_i$$

for $1 \leq i \leq n$.

- Thus

$$\sum_{i=1}^n f(t_i)\Delta x_i = F(b) - F(a)$$

- By

- 6.7 Theorem

(a) If (13) holds for some P and some ε , then (13) holds (with the same ε) for every refinement of P .

(b) If (13) holds for $P = \{x_0, \dots, x_n\}$ and if s_i, t_i are arbitrary points in $[x_{i-1}, x_i]$, then

$$\sum_{i=1}^n |f(s_i) - f(t_i)| \Delta \alpha_i < \varepsilon.$$

- (c) If $f \in \mathcal{R}(\alpha)$ and the hypotheses of (b) hold, then

$$\left| \sum_{i=1}^n f(t_i) \Delta \alpha_i - \int_a^b f \, d\alpha \right| < \varepsilon.$$

Proof Theorem 6.4 implies (a). Under the assumptions made in (b), both $f(s_i)$ and $f(t_i)$ lie in $[m_i, M_i]$, so that $|f(s_i) - f(t_i)| \leq M_i - m_i$. Thus

$$\sum_{i=1}^n |f(s_i) - f(t_i)| \Delta \alpha_i \leq U(P, f, \alpha) - L(P, f, \alpha),$$

which proves (b). The obvious inequalities

$$L(P, f, \alpha) \leq \sum f(t_i) \Delta \alpha_i \leq U(P, f, \alpha)$$

and

$$L(P, f, \alpha) \leq \int f \, d\alpha \leq U(P, f, \alpha)$$

prove (c).

we have

$$\left| F(b) - F(a) - \int_{[a,b]} f \right| < \epsilon$$

Current note has 0 direct children and 0 total descendants.

- [stem](#)

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• [int](#)

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And it has 4 siblings.

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